

LAN PROPERTY FOR AN ERGODIC ORNSTEIN-UHLENBECK PROCESS WITH POISSON JUMPS

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ABSTRACT. In this paper, we consider an ergodic Ornstein-Uhlenbeck process with jumps driven by a Brownian motion and a compensated Poisson process, whose drift and diffusion coefficients as well as its jump intensity depend on unknown parameters. Considering the process discretely observed at high frequency, we derive the local asymptotic normality property. To obtain this result, Malliavin calculus and Girsanov's theorem are applied to write the log-likelihood ratio in terms of sums of conditional expectations, for which a central limit theorem for triangular arrays can be applied.

1. INTRODUCTION

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which will be specified later on, we consider an Ornstein-Uhlenbeck (O-U) process with Poisson jumps $X^{\theta, \sigma, \lambda} = (X_t^{\theta, \sigma, \lambda})_{t \geq 0}$ in \mathbb{R} defined by

$$X_t^{\theta, \sigma, \lambda} = x_0 - \theta \int_0^t X_s^{\theta, \sigma, \lambda} ds + \sigma B_t + N_t - \lambda t, \quad (1.1)$$

where $x_0 \in \mathbb{R}$, $B = (B_t)_{t \geq 0}$ is a Brownian motion, $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$ independent of B , and $(\tilde{N}_t^\lambda)_{t \geq 0}$ denotes the compensated Poisson process $\tilde{N}_t^\lambda := N_t - \lambda t$. The parameters $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ are unknown, and Θ, Σ and Λ are closed intervals of \mathbb{R}_+^* , where $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$. Let $\{\hat{\mathcal{F}}_t\}_{t \geq 0}$ denote the natural filtration generated by B and N . We denote by $\mathbb{P}^{\theta, \sigma, \lambda}$ the probability law induced by the $\{\hat{\mathcal{F}}_t\}_{t \geq 0}$ -adapted càdlàg stochastic process $X^{\theta, \sigma, \lambda}$, and by $\mathbb{E}^{\theta, \sigma, \lambda}$ the expectation with respect to (w.r.t.) $\mathbb{P}^{\theta, \sigma, \lambda}$. Let $\xrightarrow{\mathbb{P}^{\theta, \sigma, \lambda}}$ and $\xrightarrow{\mathcal{L}(\mathbb{P}^{\theta, \sigma, \lambda})}$ denote the convergence in $\mathbb{P}^{\theta, \sigma, \lambda}$ -probability and in $\mathbb{P}^{\theta, \sigma, \lambda}$ -law, respectively.

Since $\theta > 0$, $X^{\theta, \sigma, \lambda}$ is ergodic with a unique invariant probability measure $\pi_{\theta, \sigma, \lambda}(dx)$. Furthermore, $\pi_{\theta, \sigma, \lambda}(dx)$ can be calculated explicitly (see [19, Theorem 17.5 and Corollary 17.9]), and satisfies $\int_{\mathbb{R}} |x|^p \pi_{\theta, \sigma, \lambda}(dx) < \infty$, for any $p \geq 0$. By Itô's formula, $X^{\theta, \sigma, \lambda}$ is given by

$$X_t^{\theta, \sigma, \lambda} = x_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dB_s + \int_0^t e^{-\theta(t-s)} d(N_s - \lambda s). \quad (1.2)$$

In this paper, we are interested in the estimation of the parameters $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ when the observation of the process $X^{\theta, \sigma, \lambda}$ is the discretized path $(X_{k\Delta_n}^{\theta, \sigma, \lambda})_{0 \leq k \leq n}$, where $\{\Delta_n\}_{n \in \mathbb{N}^*}$ is a sequence of time-step sizes. We denote by $\mathbb{P}_n^{\theta, \sigma, \lambda}$ the probability law of the random vector

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$(X_{k\Delta_n}^{\theta,\sigma,\lambda})_{0 \leq k \leq n}$. Let us now introduce the fundamental concept in parametric estimation called the local asymptotic normality (LAN) property which was introduced by Le Cam [14] and extended by Jeganathan [10] to the local asymptotic mixed normality (LAMN) property. In our context, we say that the sequence $\{P_n^{\theta,\sigma,\lambda}\}_{(\theta,\sigma,\lambda) \in \Theta \times \Sigma \times \Lambda}$ of parametric statistical models has the LAN property for the likelihood at $(\theta_0, \sigma_0, \lambda_0) \in \Theta \times \Sigma \times \Lambda$ with rate of convergence $(\sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n\Delta_n})$ and with covariance matrix $\Gamma(\theta_0, \sigma_0, \lambda_0)$ if for any $z = (u, v, w) \in \mathbb{R}^3$,

$$\log \frac{dP_n^{\theta_0 + \frac{u}{\sqrt{n\Delta_n}}, \sigma_0 + \frac{v}{\sqrt{n}}, \lambda_0 + \frac{w}{\sqrt{n\Delta_n}}}}{dP_n^{\theta_0, \sigma_0, \lambda_0}} \left((X_{k\Delta_n}^{\theta_0, \sigma_0, \lambda_0})_{0 \leq k \leq n} \right) \xrightarrow{\mathcal{L}(P^{\theta_0, \sigma_0, \lambda_0})} z^* \mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0)) - \frac{1}{2} z^* \Gamma(\theta_0, \sigma_0, \lambda_0) z,$$

as $n \rightarrow \infty$, where $\mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0))$ is a centered \mathbb{R}^3 -valued Gaussian vector with covariance matrix $\Gamma(\theta_0, \sigma_0, \lambda_0)$, and $*$ denotes the transpose. Here, $\Gamma(\theta_0, \sigma_0, \lambda_0)$ is called the asymptotic Fisher information matrix of the parametric statistical models.

Recall that one can define the notion of asymptotic efficiency of estimators in terms of classical Cramér-Rao lower bound. Another approach to define the asymptotic efficiency of the estimators is to study the lower bound for the asymptotic variance of the estimators via a convolution theorem. In fact, this problem is closely related to the consequence of the LAN property. Precisely, when the LAN property holds at $(\theta_0, \sigma_0, \lambda_0)$ with nonsingular $\Gamma(\theta_0, \sigma_0, \lambda_0)$, convolution and minimax theorems can be applied (see [8], [15]). Indeed, on one hand, the convolution theorem suggests the notion of asymptotically efficient estimators in terms of minimal asymptotic variance. That is, a sequence $\{(\hat{\theta}_n, \hat{\sigma}_n, \hat{\lambda}_n)\}_{n \in \mathbb{N}^*}$ of unbiased estimators of the parameter $(\theta_0, \sigma_0, \lambda_0)$ is said to be asymptotically efficient at $(\theta_0, \sigma_0, \lambda_0)$ in the sense of Hájek-Le Cam convolution theorem if as $n \rightarrow \infty$,

$$\varphi_n^{-1}(\theta_0, \sigma_0, \lambda_0) \left((\hat{\theta}_n, \hat{\sigma}_n, \hat{\lambda}_n) - (\theta_0, \sigma_0, \lambda_0) \right)^* \xrightarrow{\mathcal{L}(P^{\theta_0, \sigma_0, \lambda_0})} \mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0)^{-1}),$$

where $\varphi_n(\theta_0, \sigma_0, \lambda_0) := \text{diag}(\sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n\Delta_n})$ is the diagonal matrix. Notice that a sequence of estimators which is asymptotically efficient in the sense of Hájek-Le Cam convolution theorem achieves asymptotically the Cramér-Rao lower bound $\Gamma(\theta_0, \sigma_0, \lambda_0)^{-1}$ for the estimation variance. On the other hand, the minimax theorem states that $\Gamma(\theta_0, \sigma_0, \lambda_0)^{-1}$ gives the lower bound for the asymptotic variance of any sequence of unbiased estimators. This justifies the importance of the LAN property in parametric estimation.

The question of asymptotic efficiency of estimators for ergodic diffusions with jumps was solved e.g. in [16, 20]. The estimators in [20] are constructed from a contrast function which is based on a discretization of the likelihood function associated to the continuous observations of an ergodic diffusion with jumps whose unknown parameters appear in the drift and diffusion coefficients and in the jump coefficient as well. In [16], to estimate the drift parameter of the Ornstein-Uhlenbeck processes driven by a Lévy process, the estimators are constructed from a discretization of the time-continuous maximum likelihood estimators. These estimators are asymptotically efficient in the sense of Hájek-Le Cam convolution theorem (see [20, Theorem 2.1, Remark 2.2], [16, Theorem 3.5, Remark 3.6]).

Initiated by Gobet [6], Malliavin calculus has recently been used to analyze the log-likelihood ratio of the discrete observation of diffusion processes (with jumps). Concretely,

Gobet [6], [7] obtained the LAMN and LAN properties respectively for multidimensional elliptic diffusions and ergodic diffusions on the basis of discrete observations at high frequency. In the presence of jumps, see e.g. Aït-Sahalia and Jacod [1], Kawai [11], Clément *et al.* [3, 4], Kohatsu-Higa *et al.* [12, 13]. It seems that the validity of the LAN property for general stochastic differential equations (SDEs) with jumps whose unknown parameters appear in the drift and diffusion coefficients and in the jump component as well, has never been addressed in the literature. This is because the behaviour of the transition density, which is not explicit in general, changes strongly due to the presence of jumps. In this paper, we focus on the study of the global parameter estimation for the O-U process (1.1) whose unbounded drift and diffusion coefficients as well as its jump intensity depend on unknown parameters. Recall that in [12], the global parameter estimation is considered for a Lévy process with bounded drift coefficient. Recently, [13] has studied a multidimensional SDE with jumps whose unknown parameter appears only in the drift coefficient. Kawai [11] has dealt with an O-U process with jumps whose jump component does not depend on the unknown parameter.

In physics, the O-U process describes the velocity of a massive Brownian particle under the influence of friction. In mathematical finance, it is widely used to model the evolution of interest rates, currency exchange rates (see e.g. [2, 5]). The O-U process has a density which is highly tractable, the proof of our result, however, seems to be quite complicated. One of the motivations of writing this paper is to present a methodology which can be used to derive the LAN property for general SDEs with jumps with global parameters.

The objective of this paper is to derive the LAN property for $X^{\theta, \sigma, \lambda}$ based on discrete observations. Towards this aim, our result uses the Girsanov's theorem and the Malliavin calculus w.r.t. the Brownian motion to derive an explicit expression for the logarithm derivatives of the transition density. Moreover, when treating the negligible terms, one difficulty comes from the fact that the conditional expectations are computed under the probability measures $P^{\theta(\ell), \sigma_0, \lambda_0}$ (or $P^{\theta_n, \sigma(\ell), \lambda_n}$, or $P^{\theta_n, \sigma_0, \lambda(\ell)}$) which come from the application of the Malliavin calculus and the Girsanov's theorem, whereas the convergence is considered under $P^{\theta_0, \sigma_0, \lambda_0} \neq P^{\theta(\ell), \sigma_0, \lambda_0}$ (or $\neq P^{\theta_n, \sigma(\ell), \lambda_n}$, or $\neq P^{\theta_n, \sigma_0, \lambda(\ell)}$). To simplify the exposition, these parameters and corresponding measures will be specified in Section 2 and Subsection 4.1. To this end, when two corresponding diffusion parameters are the same as σ_0 , the Girsanov's theorem in Lemma 3.2 can be applied to change the measures. When they are different ($\sigma(\ell) \neq \sigma_0$), we need to condition on all the possible number of jumps and jump times of the Poisson process occurring on each interval $[t_k, t_{k+1}]$ in order that the change of measures can be done via the transition densities conditioned on the number of jumps and jump times (see Lemmas 6.1, 4.7 and 4.8). As a consequence, the Gaussian type expression for these transition densities is strongly used. Moreover, our proof uses the large deviation type estimates which should be understood in the sense that we study the behaviour of rare events whose decreasing rate is exponential and polynomial (see Lemma 6.6). For this, the key argument consists in conditioning on the number of jumps within the conditional expectation and in the conditioning random variable, which expresses the transition density conditioned on the number of jumps. When these two conditionings relate to different jumps, we may use a large deviation principle in the estimate. When they are equal, we use the complementary set in order to apply the large deviation principle. Within all these arguments, the Gaussian type expression for the transition density conditioned on the jump structure is again strongly used. Let us mention here that the exponential decay comes from the fact that the study of the asymptotic behaviour of the transition density leads us to study the behaviour of the

transition density under the additional condition on the number of jumps which has to be compared with another transition density with a different number of jumps. This will be seen in the estimate of the terms $M_{0,1,1,p}^{\theta,\sigma,\lambda}$ and $M_{1,1,0,1,p}^{\theta,\sigma,\lambda}$ in the proof of Lemma 6.6.

This paper is organized as follows. In Section 2, we state our main result in Theorem 2.1. Section 3 introduces preliminary results needed for the proof of Theorem 2.1, such as Girsanov's theorem, explicit expressions for the logarithm derivatives of the transition density using Malliavin calculus and Girsanov's theorem, and decompositions of the Skorohod integral. We prove our main result in Section 4. Section 5 is devoted to the conclusion and discussion of some future work. The proofs of some technical propositions and lemmas are presented in Section 6, where some crucial estimates related to the transition density conditioned on the jump structure and the large deviation type estimates are derived.

2. MAIN RESULT

For fixed $(\theta_0, \sigma_0, \lambda_0) \in \Theta \times \Sigma \times \Lambda$ and $n \geq 1$, we consider a discrete observation scheme at equidistant times $t_k = k\Delta_n$, $k \in \{0, \dots, n\}$ of the process $X^{\theta_0, \sigma_0, \lambda_0}$, which is denoted by $X^n = (X_{t_0}, X_{t_1}, \dots, X_{t_n})$, where $\Delta_n \leq 1$ for all $n \geq 1$. We assume that the high-frequency, infinite horizon and decreasing rate conditions hold. That is, $\Delta_n \rightarrow 0$, $n\Delta_n \rightarrow \infty$, and $n\Delta_n^2 \rightarrow \infty$ as $n \rightarrow \infty$.

Given the process $X^{\theta, \sigma, \lambda} = (X_t^{\theta, \sigma, \lambda})_{t \geq 0}$, we denote by $p_n(\cdot; (\theta, \sigma, \lambda))$ the density of the random vector $(X_{t_0}^{\theta, \sigma, \lambda}, X_{t_1}^{\theta, \sigma, \lambda}, \dots, X_{t_n}^{\theta, \sigma, \lambda})$. In particular, $p_n(\cdot; (\theta_0, \sigma_0, \lambda_0))$ denotes the density of the discrete observation X^n . For $(u, v, w) \in \mathbb{R}^3$, we set $\theta_n := \theta_0 + \frac{u}{\sqrt{n\Delta_n}}$, $\sigma_n := \sigma_0 + \frac{v}{\sqrt{n}}$, $\lambda_n := \lambda_0 + \frac{w}{\sqrt{n\Delta_n}}$.

The main result of this paper is the following LAN property.

Theorem 2.1. *The LAN property holds for the likelihood at $(\theta_0, \sigma_0, \lambda_0)$ with rate of convergence $(\sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n\Delta_n})$ and asymptotic Fisher information matrix $\Gamma(\theta_0, \sigma_0, \lambda_0)$. That is, for all $z = (u, v, w) \in \mathbb{R}^3$, as $n \rightarrow \infty$,*

$$\log \frac{p_n(X^n; (\theta_n, \sigma_n, \lambda_n))}{p_n(X^n; (\theta_0, \sigma_0, \lambda_0))} \xrightarrow{\mathcal{L}(\mathbb{P}^{\theta_0, \sigma_0, \lambda_0})} z^* \mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0)) - \frac{1}{2} z^* \Gamma(\theta_0, \sigma_0, \lambda_0) z,$$

where $\mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0))$ is a centered \mathbb{R}^3 -valued Gaussian vector with covariance matrix

$$\Gamma(\theta_0, \sigma_0, \lambda_0) := \frac{1}{\sigma_0^2} \begin{pmatrix} \frac{1}{2\theta_0} (\sigma_0^2 + 1) & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 + \frac{\sigma_0^2}{\lambda_0} \end{pmatrix}.$$

Remark 2.2. *Our result can be viewed as an extension of the result obtained by Gobet [7] which is only valid for the one-dimensional linear case with the presence of jumps. Moreover, when the Poisson component is degenerate ($\lambda = 0$), we recover the asymptotic Fisher information matrix of ergodic Ornstein-Uhlenbeck process without jumps (see Gobet [7, Theorem 4.1]).*

3. PRELIMINARIES

In this section, we introduce some preliminary results needed for the proof of Theorem 2.1. Towards this aim, we consider the canonical filtered probability spaces $(\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}_{t \geq 0}, \mathbb{P}^i)$,

$i \in \{1, \dots, 4\}$, associated respectively to each of the four processes B, N, W and M , where $W = (W_t)_{t \geq 0}$ is a Brownian motion, and $M = (M_t)_{t \geq 0}$ is a Poisson process with intensity λ . The processes (B, N, W, M) are mutually independent. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be the product filtered probability space of these four canonical spaces. We set $\widehat{\Omega} = \Omega^1 \times \Omega^2$, $\widehat{\mathcal{F}} = \mathcal{F}^1 \otimes \mathcal{F}^2$, $\widehat{\mathbb{P}} = \mathbb{P}^1 \otimes \mathbb{P}^2$, $\widehat{\mathcal{F}}_t = \mathcal{F}_t^1 \otimes \mathcal{F}_t^2$, $\widetilde{\Omega} = \Omega^3 \times \Omega^4$, $\widetilde{\mathcal{F}} = \mathcal{F}^3 \otimes \mathcal{F}^4$, $\widetilde{\mathbb{P}} = \mathbb{P}^3 \otimes \mathbb{P}^4$, and $\widetilde{\mathcal{F}}_t = \mathcal{F}_t^3 \otimes \mathcal{F}_t^4$. Then, $\Omega = \widehat{\Omega} \times \widetilde{\Omega}$, $\mathcal{F} = \widehat{\mathcal{F}} \otimes \widetilde{\mathcal{F}}$, $\mathbb{P} = \widehat{\mathbb{P}} \otimes \widetilde{\mathbb{P}}$, $\mathcal{F}_t = \widehat{\mathcal{F}}_t \otimes \widetilde{\mathcal{F}}_t$, and $\mathbb{E} = \widehat{\mathbb{E}} \otimes \widetilde{\mathbb{E}}$, where \mathbb{E} , $\widehat{\mathbb{E}}$, $\widetilde{\mathbb{E}}$ denote the expectation w.r.t. \mathbb{P} , $\widehat{\mathbb{P}}$ and $\widetilde{\mathbb{P}}$, respectively.

In order to deal with the likelihood ratio in Theorem 2.1, we use the following decomposition

$$\begin{aligned} \log \frac{p_n(X^n; (\theta_n, \sigma_n, \lambda_n))}{p_n(X^n; (\theta_0, \sigma_0, \lambda_0))} &= \log \frac{p_n(X^n; (\theta_n, \sigma_n, \lambda_n))}{p_n(X^n; (\theta_n, \sigma_0, \lambda_n))} + \log \frac{p_n(X^n; (\theta_n, \sigma_0, \lambda_n))}{p_n(X^n; (\theta_n, \sigma_0, \lambda_0))} \\ &\quad + \log \frac{p_n(X^n; (\theta_n, \sigma_0, \lambda_0))}{p_n(X^n; (\theta_0, \sigma_0, \lambda_0))}. \end{aligned} \quad (3.1)$$

For each of the above terms, we use the Markov property, the mean value theorem on the parameter space and then analyze each term, which leads to the logarithm derivatives of the transition density function w.r.t. the parameters. To analyze these logarithm derivatives, we start as in Gobet [6] by applying the Malliavin calculus integration by parts formula on each $[t_k, t_{k+1}]$ to derive an explicit expression for the logarithm derivatives of the transition density w.r.t. θ and σ . To avoid confusion with the observed process $X^{\theta, \sigma, \lambda}$, we introduce an extra probabilistic representation of $X^{\theta, \sigma, \lambda}$ for which the Malliavin calculus is applied. Explicitly, we consider on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the flow $Y^{\theta, \sigma, \lambda}(s, x) = (Y_t^{\theta, \sigma, \lambda}(s, x), t \geq s)$, $x \in \mathbb{R}$ on the time interval $[s, \infty)$ and with initial condition $Y_s^{\theta, \sigma, \lambda}(s, x) = x$ satisfying

$$Y_t^{\theta, \sigma, \lambda}(s, x) = x - \theta \int_s^t Y_u^{\theta, \sigma, \lambda}(s, x) du + \sigma (W_t - W_s) + \widetilde{M}_t^\lambda - \widetilde{M}_s^\lambda, \quad (3.2)$$

where $(\widetilde{M}_t^\lambda)_{t \geq 0}$ is the compensated Poisson process $\widetilde{M}_t^\lambda := M_t - \lambda t$. In particular, we write $Y_t^{\theta, \sigma, \lambda} \equiv Y_t^{\theta, \sigma, \lambda}(0, x_0)$, for all $t \geq 0$. That is,

$$Y_t^{\theta, \sigma, \lambda} = x_0 - \theta \int_0^t Y_s^{\theta, \sigma, \lambda} ds + \sigma W_t + \widetilde{M}_t^\lambda. \quad (3.3)$$

We will apply the Malliavin calculus on the Wiener space induced by W . Let D and δ denote the Malliavin derivative and the Skorohod integral w.r.t. W on each interval $[t_k, t_{k+1}]$. We denote by $\mathbb{D}^{1,2}$ the space of random variables differentiable w.r.t. W in the sense of Malliavin, and by $\text{Dom } \delta$ the domain of δ . The Malliavin calculus adapted to our model is discussed e.g. in [5, 18]. See Nualart [17] for a detailed exposition of the classical Malliavin calculus and the notations we use in this paper. For any $k \in \{0, \dots, n-1\}$, $(Y_t^{\theta, \sigma, \lambda}(t_k, x), t \in [t_k, t_{k+1}])$ is differentiable w.r.t. x, θ, σ , and we denote $(\partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x), t \in [t_k, t_{k+1}])$, $(\partial_\theta Y_t^{\theta, \sigma, \lambda}(t_k, x), t \in [t_k, t_{k+1}])$ and $(\partial_\sigma Y_t^{\theta, \sigma, \lambda}(t_k, x), t \in [t_k, t_{k+1}])$, respectively. These processes are the solutions to the linear equations

$$\partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x) = 1 - \theta \int_{t_k}^t \partial_x Y_u^{\theta, \sigma, \lambda}(t_k, x) du, \quad (3.4)$$

$$\partial_\theta Y_t^{\theta, \sigma, \lambda}(t_k, x) = - \int_{t_k}^t \left(Y_u^{\theta, \sigma, \lambda}(t_k, x) + \theta \partial_\theta Y_u^{\theta, \sigma, \lambda}(t_k, x) \right) du, \quad (3.5)$$

$$\partial_\sigma Y_t^{\theta,\sigma,\lambda}(t_k, x) = -\theta \int_{t_k}^t \partial_\sigma Y_u^{\theta,\sigma,\lambda}(t_k, x) du + W_t - W_{t_k}. \quad (3.6)$$

For any $t \in [t_k, t_{k+1}]$, the random variables $Y_t^{\theta,\sigma,\lambda}(t_k, x)$, $\partial_x Y_t^{\theta,\sigma,\lambda}(t_k, x)$, $(\partial_x Y_t^{\theta,\sigma,\lambda}(t_k, x))^{-1}$, $\partial_\theta Y_t^{\theta,\sigma,\lambda}(t_k, x)$ and $\partial_\sigma Y_t^{\theta,\sigma,\lambda}(t_k, x)$ belong to $\mathbb{D}^{1,2}$. Furthermore, by [18, Proposition 7], the Malliavin derivative $D_s Y_t^{\theta,\sigma,\lambda}(t_k, x)$ is given by

$$D_s Y_t^{\theta,\sigma,\lambda}(t_k, x) = \sigma \partial_x Y_t^{\theta,\sigma,\lambda}(t_k, x) (\partial_x Y_s^{\theta,\sigma,\lambda}(t_k, x))^{-1} \mathbf{1}_{[t_k, t]}(s).$$

For all $A \in \tilde{\mathcal{F}}$ and $x \in \mathbb{R}$, we set $\tilde{P}_x^{\theta,\sigma,\lambda}(A) = \tilde{E}[\mathbf{1}_A | Y_{t_k}^{\theta,\sigma,\lambda} = x]$. We denote by $\tilde{E}_x^{\theta,\sigma,\lambda}$ the expectation w.r.t. $\tilde{P}_x^{\theta,\sigma,\lambda}$. That is, for all $\tilde{\mathcal{F}}$ -measurable random variable V , we have that $\tilde{E}_x^{\theta,\sigma,\lambda}[V] = \tilde{E}[V | Y_{t_k}^{\theta,\sigma,\lambda} = x]$.

For any $t > s$, the law of $Y_t^{\theta,\sigma,\lambda}$ conditioned on $Y_s^{\theta,\sigma,\lambda} = x$ admits a positive transition density $p^{\theta,\sigma,\lambda}(t - s, x, y)$, which is differentiable w.r.t. θ , σ , and λ . As a consequence of [6, Proposition 4.1], we have the following explicit expression for the logarithm derivatives of the transition density w.r.t. θ and σ in terms of a conditional expectation.

Proposition 3.1. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $k \in \{0, \dots, n-1\}$, $\beta \in \{\theta, \sigma\}$, and $x, y \in \mathbb{R}$,*

$$\frac{\partial_\beta p^{\theta,\sigma,\lambda}}{p^{\theta,\sigma,\lambda}}(\Delta_n, x, y) = \frac{1}{\Delta_n} \tilde{E}_x^{\theta,\sigma,\lambda} \left[\delta \left(\partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) | Y_{t_{k+1}}^{\theta,\sigma,\lambda} = y \right],$$

where $U^{\theta,\sigma,\lambda}(t_k, x) := (U_t^{\theta,\sigma,\lambda}(t_k, x), t \in [t_k, t_{k+1}])$ with $U_t^{\theta,\sigma,\lambda}(t_k, x) := (D_t Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x))^{-1}$.

Proof. Let f be a continuously differentiable function with compact support. The chain rule of the Malliavin calculus implies that $f'(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x)) = D_t(f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x))) U_t^{\theta,\sigma,\lambda}(t_k, x)$, for all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ and $t \in [t_k, t_{k+1}]$, where $U_t^{\theta,\sigma,\lambda}(t_k, x) := (D_t Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x))^{-1}$. Moreover, it is easy to check that $\partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \in \mathbb{D}^{1,2}(H)$, where $H = L^2([t_k, t_{k+1}], \mathbb{R})$. By [17, Proposition 1.3.1], we have that $\partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \in \text{Dom } \delta$. Then, using the Malliavin calculus integration by parts formula on $[t_k, t_{k+1}]$, we get that

$$\begin{aligned} \partial_\beta \tilde{E} \left[f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x)) \right] &= \tilde{E} \left[f'(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x)) \partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) \right] \\ &= \frac{1}{\Delta_n} \tilde{E} \left[\int_{t_k}^{t_{k+1}} D_t(f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x))) U_t^{\theta,\sigma,\lambda}(t_k, x) \partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) dt \right] \\ &= \frac{1}{\Delta_n} \tilde{E} \left[f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x)) \delta \left(\partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) \right]. \end{aligned}$$

Note that $\delta(V) \equiv \delta(V \mathbf{1}_{[t_k, t_{k+1}]}(\cdot))$ for any $V \in \text{Dom } \delta$. Using the fact that $p^{\theta,\sigma,\lambda}(\Delta_n, x, y)$ and $\partial_\beta p^{\theta,\sigma,\lambda}(\Delta_n, x, y)$ are continuous for (y, β) , the stochastic flow property $Y_t^{\theta,\sigma,\lambda} = Y_t^{\theta,\sigma,\lambda}(s, Y_s^{\theta,\sigma,\lambda})$ for all $0 \leq s \leq t$, and the Markov property of diffusion processes, we have that

$$\partial_\beta \tilde{E} \left[f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x)) \right] = \int_{\mathbb{R}} f(y) \partial_\beta p^{\theta,\sigma,\lambda}(\Delta_n, x, y) dy,$$

and

$$\begin{aligned} &\tilde{E} \left[f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x)) \delta \left(\partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) \right] \\ &= \tilde{E} \left[f(Y_{t_{k+1}}^{\theta,\sigma,\lambda}) \delta \left(\partial_\beta Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) | Y_{t_k}^{\theta,\sigma,\lambda} = x \right] \end{aligned}$$

$$= \int_{\mathbb{R}} f(y) \tilde{\mathbb{E}} \left[\delta \left(\partial_{\beta} Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) \middle| Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y, Y_{t_k}^{\theta, \sigma, \lambda} = x \right] p^{\theta, \sigma, \lambda}(\Delta_n, x, y) dy,$$

which gives the desired result. \square

For all $k \in \{0, \dots, n-1\}$, we set $\Delta V_{t_{k+1}} := V_{t_{k+1}} - V_{t_k}$, for $V \in \{X^{\theta_0, \sigma_0, \lambda_0}, B, N, W, M\}$. We next recall Girsanov's theorem on each interval $[t_k, t_{k+1}]$.

Lemma 3.2. *For all $\theta, \theta_1, \lambda, \lambda_1, \sigma \in \mathbb{R}_+^*$ and $k \in \{0, \dots, n-1\}$, define the measure*

$$\hat{Q}_k^{\theta_1, \lambda_1, \theta, \lambda, \sigma}(A) := \hat{\mathbb{E}} \left[\mathbf{1}_A e^{\int_{t_k}^{t_{k+1}} \frac{(\theta - \theta_1)X_s + \lambda - \lambda_1}{\sigma} dB_s + \frac{1}{2} \int_{t_k}^{t_{k+1}} \left(\frac{(\theta - \theta_1)X_s + \lambda - \lambda_1}{\sigma} \right)^2 ds - \Delta N_{t_{k+1}} \log \frac{\lambda}{\lambda_1} + (\lambda - \lambda_1) \Delta_n} \right],$$

for all $A \in \hat{\mathcal{F}}$. Then $\hat{Q}_k^{\theta_1, \lambda_1, \theta, \lambda, \sigma}$ is a probability measure and under $\hat{Q}_k^{\theta_1, \lambda_1, \theta, \lambda, \sigma}$, the process $(B_t^{\hat{Q}_k^{\theta_1, \lambda_1, \theta, \lambda, \sigma}} := B_t - \int_{t_k}^t \frac{(\theta - \theta_1)X_s + \lambda - \lambda_1}{\sigma} ds, t \in [t_k, t_{k+1}])$ is a Brownian motion, and $(N_t, t \in [t_k, t_{k+1}])$ is a Poisson process with intensity λ_1 . Moreover, all these statements remain valid for the measure $\tilde{Q}_k^{\theta_1, \lambda_1, \theta, \lambda, \sigma}$ defined by, for all $A \in \tilde{\mathcal{F}}$,

$$\tilde{Q}_k^{\theta_1, \lambda_1, \theta, \lambda, \sigma}(A) := \tilde{\mathbb{E}} \left[\mathbf{1}_A e^{\int_{t_k}^{t_{k+1}} \frac{(\theta - \theta_1)Y_s + \lambda - \lambda_1}{\sigma} dW_s + \frac{1}{2} \int_{t_k}^{t_{k+1}} \left(\frac{(\theta - \theta_1)Y_s + \lambda - \lambda_1}{\sigma} \right)^2 ds - \Delta M_{t_{k+1}} \log \frac{\lambda}{\lambda_1} + (\lambda - \lambda_1) \Delta_n} \right].$$

As a consequence, we derive the following explicit expression for the logarithm derivative of the transition density w.r.t. λ in the form of conditional expectation.

Proposition 3.3. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $k \in \{0, \dots, n-1\}$, and $x, y \in \mathbb{R}$,*

$$\frac{\partial_{\lambda} p^{\theta, \sigma, \lambda}}{p^{\theta, \sigma, \lambda}}(\Delta_n, x, y) = \tilde{\mathbb{E}}_x^{\theta, \sigma, \lambda} \left[-\frac{\Delta W_{t_{k+1}}}{\sigma} + \frac{\tilde{M}_{t_{k+1}}^{\lambda} - \tilde{M}_{t_k}^{\lambda}}{\lambda} \middle| Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y \right].$$

Proof. Let f be a continuously differentiable bounded function. Girsanov's theorem yields

$$\tilde{\mathbb{E}} \left[f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \right] = \tilde{\mathbb{E}} \left[f(Y_{t_{k+1}}^{\theta, \sigma, \lambda_1}(t_k, x)) \frac{d\tilde{\mathbb{P}}}{d\tilde{Q}_k^{\theta, \lambda_1, \theta, \lambda, \sigma}} \right].$$

Taking the derivative w.r.t. λ in both hand sides of this equality and using Lemma 3.2,

$$\begin{aligned} \partial_{\lambda} \tilde{\mathbb{E}} \left[f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \right] &= \tilde{\mathbb{E}} \left[f(Y_{t_{k+1}}^{\theta, \sigma, \lambda_1}(t_k, x)) \partial_{\lambda} \left(\frac{d\tilde{\mathbb{P}}}{d\tilde{Q}_k^{\theta, \lambda_1, \theta, \lambda, \sigma}} \right) \right] \\ &= \tilde{\mathbb{E}} \left[f(Y_{t_{k+1}}^{\theta, \sigma, \lambda_1}(t_k, x)) \left(-\frac{\Delta W_{t_{k+1}}}{\sigma} - \frac{\lambda - \lambda_1}{\sigma^2} \Delta_n + \frac{\tilde{M}_{t_{k+1}}^{\lambda} - \tilde{M}_{t_k}^{\lambda}}{\lambda} \right) \frac{d\tilde{\mathbb{P}}}{d\tilde{Q}_k^{\theta, \lambda_1, \theta, \lambda, \sigma}} \right] \\ &= \tilde{\mathbb{E}} \left[f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \left(-\frac{\Delta W_{t_{k+1}}}{\sigma} + \frac{\tilde{M}_{t_{k+1}}^{\lambda} - \tilde{M}_{t_k}^{\lambda}}{\lambda} \right) \right]. \end{aligned}$$

Then proceeding similarly as in the proof of Proposition 3.1, the desired result follows. \square

From the decomposition 3.1, the Markov property and Propositions 3.1, 3.3, the log-likelihood ratio will be represented as sums of conditional expectations involving Skorohod integrals. Therefore, in order to control the convergence of the log-likelihood ratio, one needs to have an appropriate stochastic expansion of the Skorohod integrals $\delta(\partial_{\beta} Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x))$ appearing in Proposition 3.1, with $\beta \in \{\theta, \sigma\}$, and of the conditional expectations appearing

in Proposition 3.3. In fact, these random variables can be decomposed in terms of some function $g(\theta, \sigma, \lambda, \Delta_n, Y_{t_k}^{\theta, \sigma, \lambda}, Y_{t_{k+1}}^{\theta, \sigma, \lambda})$ and remainder terms. Then $g(\theta, \sigma, \lambda, \Delta_n, Y_{t_k}^{\theta, \sigma, \lambda}, Y_{t_{k+1}}^{\theta, \sigma, \lambda})$ whose conditional expectation can be computed easily corresponds to the main terms that will contribute to the limit of the log-likelihood ratio. Some remainder terms which have to be centered random variables (see (3.11) and (3.14)) will have no contribution in the limit of the log-likelihood ratio. These facts will be seen in Subsections 4.1-4.3. Consequently, we have the following decompositions of the aforementioned Skorohod integrals.

Lemma 3.4. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $k \in \{0, \dots, n-1\}$, and $x \in \mathbb{R}$,*

$$\begin{aligned} \delta \left(\partial_\theta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) &= -\Delta_n \sigma^{-2} x \left(Y_{t_{k+1}}^{\theta, \sigma, \lambda} - Y_{t_k}^{\theta, \sigma, \lambda} + \Delta_n \theta Y_{t_k}^{\theta, \sigma, \lambda} \right) \\ &\quad + R_1^{\theta, \sigma, \lambda} + R_2^{\theta, \sigma, \lambda} + R_3^{\theta, \sigma, \lambda} - R_4^{\theta, \sigma, \lambda} - R_5^{\theta, \sigma, \lambda}, \end{aligned}$$

where

$$\begin{aligned} R_1^{\theta, \sigma, \lambda} &= \frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_s \left(\frac{Y_u^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_u^{\theta, \sigma, \lambda}(t_k, x)} \right) \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) du ds, \\ R_2^{\theta, \sigma, \lambda} &= -\frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \frac{Y_s^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} ds \int_{t_k}^{t_{k+1}} \left(\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) - \partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) \right) dW_s, \\ R_3^{\theta, \sigma, \lambda} &= -\frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \left(\frac{Y_s^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} - \frac{Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)} \right) ds \int_{t_k}^{t_{k+1}} \partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) dW_s, \\ R_4^{\theta, \sigma, \lambda} &= \Delta_n \sigma^{-2} \theta x \int_{t_k}^{t_{k+1}} \left(Y_s^{\theta, \sigma, \lambda} - Y_{t_k}^{\theta, \sigma, \lambda} \right) ds, \quad R_5^{\theta, \sigma, \lambda} = -\Delta_n \sigma^{-2} x \left(\widetilde{M}_{t_{k+1}}^\lambda - \widetilde{M}_{t_k}^\lambda \right). \end{aligned}$$

Proof. From (3.4) and Itô's formula,

$$(\partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x))^{-1} = 1 + \theta \int_{t_k}^t (\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x))^{-1} ds, \quad (3.7)$$

which, together with (3.5) and Itô's formula again, implies that

$$\frac{\partial_\theta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)} = - \int_{t_k}^{t_{k+1}} \frac{Y_s^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} ds.$$

Then, using $U_t^{\theta, \sigma, \lambda}(t_k, x) = (D_t Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x))^{-1} = \sigma^{-1} (\partial_x Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x))^{-1} \partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x)$ and the product rule [17, (1.48)], we obtain that

$$\begin{aligned} \delta \left(\partial_\theta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) &= -\frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \frac{Y_s^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} ds \int_{t_k}^{t_{k+1}} \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) dW_s \\ &\quad + \frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_s \left(\frac{Y_u^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_u^{\theta, \sigma, \lambda}(t_k, x)} \right) \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) du ds. \end{aligned}$$

We next add and subtract the term $\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)$ in the second integral, and the term $\frac{Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)}$ in the first integral. This, together with $Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x$, yields

$$\delta \left(\partial_\theta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) = -\Delta_n \sigma^{-1} x \Delta W_{t_{k+1}} + R_1^{\theta, \sigma, \lambda} + R_2^{\theta, \sigma, \lambda} + R_3^{\theta, \sigma, \lambda}. \quad (3.8)$$

On the other hand, by equation (3.3) we have that

$$\sigma \Delta W_{t_{k+1}} = Y_{t_{k+1}}^{\theta, \sigma, \lambda} - Y_{t_k}^{\theta, \sigma, \lambda} + \Delta_n \theta Y_{t_k}^{\theta, \sigma, \lambda} + \theta \int_{t_k}^{t_{k+1}} \left(Y_s^{\theta, \sigma, \lambda} - Y_{t_k}^{\theta, \sigma, \lambda} \right) ds - \left(\widetilde{M}_{t_{k+1}}^\lambda - \widetilde{M}_{t_k}^\lambda \right). \quad (3.9)$$

This, together with (3.8), gives the desired result. \square

Lemma 3.5. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $k \in \{0, \dots, n-1\}$, and $x \in \mathbb{R}$,*

$$\begin{aligned} \delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) &= \frac{1}{\sigma^3} \left(Y_{t_{k+1}}^{\theta, \sigma, \lambda} - Y_{t_k}^{\theta, \sigma, \lambda} \right)^2 - \frac{\Delta_n}{\sigma} + H_1^{\theta, \sigma, \lambda} + H_2^{\theta, \sigma, \lambda} + H_3^{\theta, \sigma, \lambda} \\ &\quad - \frac{1}{\sigma^3} \left\{ \left(H_4^{\theta, \sigma, \lambda} + H_5^{\theta, \sigma, \lambda} \right)^2 + 2\sigma \Delta W_{t_{k+1}} \left(H_4^{\theta, \sigma, \lambda} + H_5^{\theta, \sigma, \lambda} \right) \right\}, \end{aligned}$$

where

$$\begin{aligned} H_1^{\theta, \sigma, \lambda} &= -\frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_s \left(\frac{1}{\partial_x Y_u^{\theta, \sigma, \lambda}(t_k, x)} \right) dW_u \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) ds, \\ H_2^{\theta, \sigma, \lambda} &= \frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \left(\frac{1}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} - \frac{1}{\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)} \right) dW_s \int_{t_k}^{t_{k+1}} \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) dW_s, \\ H_3^{\theta, \sigma, \lambda} &= \frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \frac{1}{\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)} dW_s \int_{t_k}^{t_{k+1}} \left(\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) - \partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) \right) dW_s, \\ H_4^{\theta, \sigma, \lambda} &= -\theta \int_{t_k}^{t_{k+1}} Y_s^{\theta, \sigma, \lambda} ds, \quad H_5^{\theta, \sigma, \lambda} = \widetilde{M}_{t_{k+1}}^\lambda - \widetilde{M}_{t_k}^\lambda. \end{aligned}$$

Proof. From (3.6), (3.7) and Itô's formula,

$$\frac{\partial_\sigma Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)}{\partial_x Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)} = \int_{t_k}^{t_{k+1}} \frac{1}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} dW_s.$$

Then, using $U_t^{\theta, \sigma, \lambda}(t_k, x) = \sigma^{-1} (\partial_x Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x))^{-1} \partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x)$ and the product rule [17, (1.48)], we obtain that

$$\begin{aligned} \delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) &= \frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \frac{1}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} dW_s \int_{t_k}^{t_{k+1}} \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) dW_s \\ &\quad - \frac{1}{\sigma} \int_{t_k}^{t_{k+1}} \left\{ \frac{1}{\partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x)} + \int_s^{t_{k+1}} D_s \left(\frac{1}{\partial_x Y_u^{\theta, \sigma, \lambda}(t_k, x)} \right) dW_u \right\} \partial_x Y_s^{\theta, \sigma, \lambda}(t_k, x) ds. \end{aligned}$$

We next add and subtract the term $\frac{1}{\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)}$ in the first integral, and the term $\partial_x Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x)$ in the second integral. This yields

$$\delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) = \frac{1}{\sigma} (\Delta W_{t_{k+1}})^2 - \frac{\Delta_n}{\sigma} + H_1^{\theta, \sigma, \lambda} + H_2^{\theta, \sigma, \lambda} + H_3^{\theta, \sigma, \lambda}. \quad (3.10)$$

On the other hand, by equation (3.3) we have that

$$\Delta W_{t_{k+1}} = \sigma^{-1} \left(Y_{t_{k+1}}^{\theta, \sigma, \lambda} - Y_{t_k}^{\theta, \sigma, \lambda} + \theta \int_{t_k}^{t_{k+1}} Y_s^{\theta, \sigma, \lambda} ds - \left(\widetilde{M}_{t_{k+1}}^\lambda - \widetilde{M}_{t_k}^\lambda \right) \right),$$

which, together with (3.10), gives the desired result. \square

We will use the following estimates for the solution to (3.2).

Lemma 3.6. (i) For any $p \geq 1$ and $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, there exists a constant $C_p > 0$ such that for all $k \in \{0, \dots, n-1\}$ and $t \in [t_k, t_{k+1}]$,

$$\mathbb{E} \left[\left| Y_t^{\theta, \sigma, \lambda}(t_k, x) - Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p |t - t_k|^{\frac{p}{2} \wedge 1} (1 + |x|^p).$$

(ii) For any function g defined on $\Theta \times \Sigma \times \Lambda \times \mathbb{R}$ with polynomial growth in x uniformly in $(\theta, \sigma, \lambda)$, there exist constants $C, q > 0$ such that for $k \in \{0, \dots, n-1\}$ and $t \in [t_k, t_{k+1}]$,

$$\mathbb{E} \left[\left| g(\theta, \sigma, \lambda, Y_t^{\theta, \sigma, \lambda}(t_k, x)) \right| \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C (1 + |x|^q).$$

Moreover, all these statements remain valid for $X^{\theta, \sigma, \lambda}$.

Using Gronwall's inequality, one can easily check that for any $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ and $p \geq 2$, there exist constant $C_p, q > 0$ such that for all $k \in \{0, \dots, n-1\}$ and $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} & \mathbb{E} \left[\left| \partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x) \right|^p + \frac{1}{\left| \partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x) \right|^p} + \left| \partial_\sigma Y_t^{\theta, \sigma, \lambda}(t_k, x) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \\ & + \sup_{s \in [t_k, t_{k+1}]} \mathbb{E} \left[\left| D_s Y_t^{\theta, \sigma, \lambda}(t_k, x) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p, \\ & \mathbb{E} \left[\left| \partial_\theta Y_t^{\theta, \sigma, \lambda}(t_k, x) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \\ & + \sup_{s \in [t_k, t_{k+1}]} \mathbb{E} \left[\left| D_s \left(\partial_x Y_t^{\theta, \sigma, \lambda}(t_k, x) \right) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p (1 + |x|^q). \end{aligned}$$

As a consequence, we have the following estimates, which follow easily from (3.8), (3.10), Lemma 3.6 and properties of the moments of the Brownian motion.

Lemma 3.7. For any $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ and $p \geq 2$, there exist constants $C_p, q > 0$ such that for all $k \in \{0, \dots, n-1\}$,

$$\mathbb{E} \left[R_1^{\theta, \sigma, \lambda} + R_2^{\theta, \sigma, \lambda} + R_3^{\theta, \sigma, \lambda} \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] = 0, \quad (3.11)$$

$$\mathbb{E} \left[\left| R_1^{\theta, \sigma, \lambda} + R_2^{\theta, \sigma, \lambda} + R_3^{\theta, \sigma, \lambda} \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p \Delta_n^{\frac{3p+1}{2}} (1 + |x|^q), \quad (3.12)$$

$$\mathbb{E} \left[\left| \delta \left(\partial_\theta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p \Delta_n^{\frac{3p}{2}} (1 + |x|^q), \quad (3.13)$$

$$\mathbb{E} \left[H_1^{\theta, \sigma, \lambda} + H_2^{\theta, \sigma, \lambda} + H_3^{\theta, \sigma, \lambda} \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] = 0, \quad (3.14)$$

$$\mathbb{E} \left[\left| H_1^{\theta, \sigma, \lambda} + H_2^{\theta, \sigma, \lambda} + H_3^{\theta, \sigma, \lambda} \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p \Delta_n^{p+\frac{1}{2}} (1 + |x|^q), \quad (3.15)$$

$$\mathbb{E} \left[\left| \delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) \right|^p \middle| Y_{t_k}^{\theta, \sigma, \lambda}(t_k, x) = x \right] \leq C_p \Delta_n^p (1 + |x|^q). \quad (3.16)$$

Next, we recall a discrete ergodic theorem.

Lemma 3.8. [7, Lemma 3.1] Consider a differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$, whose derivatives have polynomial growth in x . Then, as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{k=0}^{n-1} g(X_{t_k}) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} \int_{\mathbb{R}} g(x) \pi_{\theta_0, \sigma_0, \lambda_0}(dx).$$

For each $n \in \mathbb{N}^*$, let $(Z_{k,n})_{k \geq 1}$ be a sequence of random variables defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and assume that they are $\mathcal{F}_{t_{k+1}}$ -measurable, for all k .

Lemma 3.9. [7, Lemma 3.2] *Assume that as $n \rightarrow \infty$,*

$$(i) \sum_{k=0}^{n-1} \mathbb{E}[Z_{k,n} | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} 0, \quad \text{and} \quad (ii) \sum_{k=0}^{n-1} \mathbb{E}[Z_{k,n}^2 | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} 0.$$

Then as $n \rightarrow \infty$, $\sum_{k=0}^{n-1} Z_{k,n} \xrightarrow{\mathbb{P}} 0$.

Lemma 3.10. [9, Lemma 4.1] *Assume that as $n \rightarrow \infty$, $\sum_{k=0}^{n-1} \mathbb{E}[|Z_{k,n}| | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} 0$. Then as $n \rightarrow \infty$, $\sum_{k=0}^{n-1} Z_{k,n} \xrightarrow{\mathbb{P}} 0$.*

4. PROOF OF THEOREM 2.1

In this section, the proof of Theorem 2.1 will be divided into three steps. We begin deriving a stochastic expansion of the log-likelihood ratio using Propositions 3.1, 3.3 and Lemmas 3.4, 3.5. The second step treats the negligible contributions of the expansion. Finally, we apply the central limit theorem for triangular arrays in order to show the LAN property.

4.1. Expansion of the log-likelihood ratio.

Lemma 4.1. *The log-likelihood ratio at $(\theta_0, \sigma_0, \lambda_0)$ can be expressed as*

$$\begin{aligned} \log \frac{p_n(X^n; (\theta_n, \sigma_n, \lambda_n))}{p_n(X^n; (\theta_0, \sigma_0, \lambda_0))} &= \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n} + \beta_{k,n}) + \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \left\{ Z_{k,n}^{4,\ell} + Z_{k,n}^{5,\ell} \right. \\ &\quad \left. + \tilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R^{\theta(\ell), \sigma_0, \lambda_0} - R_4^{\theta(\ell), \sigma_0, \lambda_0} - R_5^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right\} d\ell + \sum_{k=0}^{n-1} (T_{k,n} - R_{k,n}) \\ &\quad + \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \tilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[H^{\theta_n, \sigma(\ell), \lambda_n} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] d\ell + \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \\ &\quad \times \left\{ (H_6 + H_7)^2 + 2\sigma_0 \Delta B_{t_{k+1}} (H_6 + H_7) - \tilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right)^2 \right. \right. \\ &\quad \left. \left. + 2\sigma(\ell) \Delta W_{t_{k+1}} \left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right) | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right\} d\ell, \end{aligned}$$

where $\theta(\ell) := \theta_0 + \frac{\ell u}{\sqrt{n\Delta_n}}$, $\sigma(\ell) := \sigma_0 + \frac{\ell v}{\sqrt{n}}$, $\lambda(\ell) := \lambda_0 + \frac{\ell w}{\sqrt{n\Delta_n}}$, and

$$\begin{aligned} \xi_{k,n} &= -\frac{u}{\sigma_0^2 \sqrt{n\Delta_n}} X_{t_k} \left(\sigma_0 \Delta B_{t_{k+1}} + \frac{u\Delta_n}{2\sqrt{n\Delta_n}} X_{t_k} \right), \\ \eta_{k,n} &= \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \left(\frac{\sigma_0^2}{\sigma(\ell)^3} (\Delta B_{t_{k+1}})^2 - \frac{\Delta_n}{\sigma(\ell)} \right) d\ell, \\ \beta_{k,n} &= -\frac{w}{\sigma_0^2 \sqrt{n\Delta_n}} \left(\sigma_0 \Delta B_{t_{k+1}} + \frac{w\Delta_n}{2\sqrt{n\Delta_n}} + \frac{u\Delta_n}{\sqrt{n\Delta_n}} X_{t_k} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{w}{\sqrt{n\Delta_n}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\frac{\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)}}{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell, \\
Z_{k,n}^{4,\ell} &= \Delta_n \sigma_0^{-2} \theta_0 X_{t_k} \int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds, \quad Z_{k,n}^{5,\ell} = -\Delta_n \sigma_0^{-2} X_{t_k} \left(\tilde{N}_{t_{k+1}}^{\lambda_0} - \tilde{N}_{t_k}^{\lambda_0} \right), \\
R^{\theta(\ell), \sigma_0, \lambda_0} &= R_1^{\theta(\ell), \sigma_0, \lambda_0} + R_2^{\theta(\ell), \sigma_0, \lambda_0} + R_3^{\theta(\ell), \sigma_0, \lambda_0}, \\
T_{k,n} &= \frac{w}{\sigma_0^2 \sqrt{n\Delta_n}} \int_0^1 \left\{ \theta_0 \int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds \right. \\
&\quad \left. - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\theta_n \int_{t_k}^{t_{k+1}} \left(Y_s^{\theta_n, \sigma_0, \lambda(\ell)} - Y_{t_k}^{\theta_n, \sigma_0, \lambda(\ell)} \right) ds | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] \right\} d\ell, \\
R_{k,n} &= \frac{w}{\sigma_0^2 \sqrt{n\Delta_n}} \int_0^1 \left(\Delta N_{t_{k+1}} - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\Delta M_{t_{k+1}} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] \right) d\ell, \\
H^{\theta_n, \sigma(\ell), \lambda_n} &= H_1^{\theta_n, \sigma(\ell), \lambda_n} + H_2^{\theta_n, \sigma(\ell), \lambda_n} + H_3^{\theta_n, \sigma(\ell), \lambda_n}, \\
H_6 &= -\theta_0 \int_{t_k}^{t_{k+1}} X_s^{\theta_0, \sigma_0, \lambda_0} ds, \quad H_7 = \tilde{N}_{t_{k+1}}^{\lambda_0} - \tilde{N}_{t_k}^{\lambda_0}.
\end{aligned}$$

Proof. Recall that the log-likelihood ratio is decomposed as in (3.1). First, using the Markov property, Proposition 3.1, Lemma 3.4 and equation (1.1) for the term $X_{t_{k+1}} - X_{t_k}$ that we obtain from the term $Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} - Y_{t_k}^{\theta(\ell), \sigma_0, \lambda_0}$ in Lemma 3.4 when taking the conditional expectation, we obtain that

$$\begin{aligned}
\log \frac{p_n(X^n; (\theta_n, \sigma_0, \lambda_0))}{p_n(X^n; (\theta_0, \sigma_0, \lambda_0))} &= \sum_{k=0}^{n-1} \log \frac{p_{\theta_n, \sigma_0, \lambda_0}}{p_{\theta_0, \sigma_0, \lambda_0}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) \\
&= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n}} \int_0^1 \frac{\partial_\theta p^{\theta(\ell), \sigma_0, \lambda_0}}{p^{\theta(\ell), \sigma_0, \lambda_0}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) d\ell \\
&= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} [\delta(\partial_\theta Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0}(t_k, X_{t_k}) U^{\theta(\ell), \sigma_0, \lambda_0}(t_k, X_{t_k})) | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}}] d\ell \\
&= \sum_{k=0}^{n-1} \xi_{k,n} + \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \left\{ Z_{k,n}^{4,\ell} + Z_{k,n}^{5,\ell} \right. \\
&\quad \left. + \tilde{E}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R^{\theta(\ell), \sigma_0, \lambda_0} - R_4^{\theta(\ell), \sigma_0, \lambda_0} - R_5^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right\} d\ell.
\end{aligned}$$

Next, using the Markov property, Proposition 3.1, Lemma 3.5 and equation (1.1) for the term $X_{t_{k+1}} - X_{t_k}$ that we obtain from the term $Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} - Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}$ in Lemma 3.5 when taking the conditional expectation, we obtain that

$$\log \frac{p_n(X^n; (\theta_n, \sigma_n, \lambda_n))}{p_n(X^n; (\theta_n, \sigma_0, \lambda_n))} = \sum_{k=0}^{n-1} \frac{v}{\sqrt{n}} \int_0^1 \frac{\partial_\sigma p^{\theta_n, \sigma(\ell), \lambda_n}}{p^{\theta_n, \sigma(\ell), \lambda_n}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) d\ell$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\delta(\partial_\sigma Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}(t_k, X_{t_k}) U^{\theta_n, \sigma(\ell), \lambda_n}(t_k, X_{t_k})) | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] d\ell \\
&= \sum_{k=0}^{n-1} \eta_{k,n} + \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[H^{\theta_n, \sigma(\ell), \lambda_n} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] d\ell \\
&\quad + \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \left\{ (H_6 + H_7)^2 + 2\sigma_0 \Delta B_{t_{k+1}} (H_6 + H_7) \right. \\
&\quad \left. - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right)^2 + 2\sigma(\ell) \Delta W_{t_{k+1}} \right. \right. \\
&\quad \left. \left. \times \left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right) | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right\} d\ell.
\end{aligned}$$

Finally, using Proposition 3.3, equation (3.9) with $(\theta_n, \sigma_0, \lambda(\ell))$ instead of $(\theta, \sigma, \lambda)$, and equation (1.1) for the term $X_{t_{k+1}} - X_{t_k}$ that we obtain from the term $Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} - Y_{t_k}^{\theta_n, \sigma_0, \lambda(\ell)}$ in (3.9) when taking the conditional expectation, we get that

$$\begin{aligned}
\log \frac{p_n(X^n; (\theta_n, \sigma_0, \lambda_n))}{p_n(X^n; (\theta_n, \sigma_0, \lambda_0))} &= \sum_{k=0}^{n-1} \frac{w}{\sqrt{n\Delta_n}} \int_0^1 \frac{\partial_\lambda p^{\theta_n, \sigma_0, \lambda(\ell)}}{p^{\theta_n, \sigma_0, \lambda(\ell)}} (\Delta_n, X_{t_k}, X_{t_{k+1}}) d\ell \\
&= \sum_{k=0}^{n-1} \frac{w}{\sqrt{n\Delta_n}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[-\frac{\Delta W_{t_{k+1}}}{\sigma_0} + \frac{\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)}}{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell \\
&= \sum_{k=0}^{n-1} \beta_{k,n} + \sum_{k=0}^{n-1} (T_{k,n} - R_{k,n}).
\end{aligned}$$

Therefore, we have shown the desired expansion of the log-likelihood ratio. \square

We show that $\xi_{k,n}, \eta_{k,n}, \beta_{k,n}$ are the terms that contribute to the limit, and all the others are negligible. In all what follows, Lemma 3.8 will be used repeatedly without being quoted.

4.2. Negligible contributions.

Lemma 4.2. *As $n \rightarrow \infty$,*

$$\sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] d\ell \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Proof. It suffices to show that conditions (i) and (ii) of Lemma 3.9 hold under the measure $P^{\theta_0, \sigma_0, \lambda_0}$. We start showing (i). Applying Girsanov's theorem and (3.11), we get that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 E \left[\tilde{E}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] | \mathcal{F}_{t_k} \right] d\ell \\
&= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 E_{\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[R^{\theta(\ell), \sigma_0, \lambda_0} \frac{d\widehat{P}}{d\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} | X_{t_k} \right] d\ell = 0,
\end{aligned}$$

where we have used the independence between $R^{\theta(\ell), \sigma_0, \lambda_0}$ and $\frac{d\hat{P}}{d\hat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}}$, together with $\mathbb{E}_{\hat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[\frac{d\hat{P}}{d\hat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} | X_{t_k} \right] = 1$. Thus the term appearing in condition (i) of Lemma 3.9 actually equals zero.

Next, applying Jensen's inequality, Girsanov's theorem, and (3.12) with $p = 2$, we obtain

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{u^2}{n\Delta_n^3} \mathbb{E} \left[\left(\int_0^1 \tilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] d\ell \right)^2 | \mathcal{F}_{t_k} \right] \\ & \leq \sum_{k=0}^{n-1} \frac{u^2}{n\Delta_n^3} \int_0^1 \mathbb{E}_{\hat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[\left(R^{\theta(\ell), \sigma_0, \lambda_0} \right)^2 \frac{d\hat{P}}{d\hat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} | X_{t_k} \right] d\ell \\ & = \sum_{k=0}^{n-1} \frac{u^2}{n\Delta_n^3} \int_0^1 \mathbb{E}_{\hat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[\left(R^{\theta(\ell), \sigma_0, \lambda_0} \right)^2 | X_{t_k} \right] d\ell \\ & \leq \frac{Cu^2\sqrt{\Delta_n}}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q), \end{aligned}$$

for some constants $C, q > 0$, which gives the desired result. \square

Lemma 4.3. *As $n \rightarrow \infty$,*

$$\sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \left(Z_{k,n}^{4,\ell} - \tilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R_4^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right) d\ell \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Proof. We rewrite

$$\frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \left(Z_{k,n}^{4,\ell} - \tilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R_4^{\theta(\ell), \sigma_0, \lambda_0} | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right) d\ell = M_{k,n,1} + M_{k,n,2},$$

where

$$\begin{aligned} M_{k,n,1} &:= -\frac{u^2}{\sigma_0^2 n \Delta_n} X_{t_k} \int_0^1 \ell \int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds d\ell, \\ M_{k,n,2} &:= \frac{u}{\sigma_0^2 \sqrt{n\Delta_n}} X_{t_k} \int_0^1 \theta(\ell) \left\{ \int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds \right. \\ & \quad \left. - \tilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[\int_{t_k}^{t_{k+1}} \left(Y_s^{\theta(\ell), \sigma_0, \lambda_0} - Y_{t_k}^{\theta(\ell), \sigma_0, \lambda_0} \right) ds | Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right\} d\ell. \end{aligned}$$

First, using Lemma 3.6(i), we get that

$$\sum_{k=0}^{n-1} \mathbb{E} \left[|M_{k,n,1}| | \mathcal{F}_{t_k} \right] \leq \frac{Cu^2\sqrt{\Delta_n}}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q),$$

for some constants $C, q > 0$. Therefore, by Lemma 3.10, $\sum_{k=0}^{n-1} M_{k,n,1} \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0$ as $n \rightarrow \infty$.

Now, we treat $M_{k,n,2}$. Using Girsanov's theorem, we have that

$$\sum_{k=0}^{n-1} \mathbb{E} \left[M_{k,n,2} | \mathcal{F}_{t_k} \right] = \frac{u}{\sigma_0^2 \sqrt{n\Delta_n}} \sum_{k=0}^{n-1} X_{t_k} \int_0^1 \theta(\ell) \left\{ \mathbb{E} \left[\int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds | X_{t_k} \right] \right.$$

$$\begin{aligned}
& - \mathbb{E}_{\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[\int_{t_k}^{t_{k+1}} \left(Y_s^{\theta(\ell), \sigma_0, \lambda_0} - Y_{t_k}^{\theta(\ell), \sigma_0, \lambda_0} \right) ds \frac{d\widehat{P}}{d\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \Big| X_{t_k} \right] \Big\} d\ell \\
& = 0,
\end{aligned}$$

where we use the independence between $\int_{t_k}^{t_{k+1}} (Y_s^{\theta(\ell), \sigma_0, \lambda_0} - Y_{t_k}^{\theta(\ell), \sigma_0, \lambda_0}) ds$ and $\frac{d\widehat{P}}{d\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}}$, together with $\mathbb{E}_{\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[\frac{d\widehat{P}}{d\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \Big| X_{t_k} \right] = 1$.

Next, we proceed as in the proof of Lemma 4.2 to get that for some constants $C, q > 0$,

$$\sum_{k=0}^{n-1} \mathbb{E} \left[M_{k,n,2}^2 \Big| \widehat{\mathcal{F}}_{t_k} \right] \leq \frac{Cu^2 \Delta_n^2}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q).$$

Therefore, by Lemma 3.9, $\sum_{k=0}^{n-1} M_{k,n,2} \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0$ as $n \rightarrow \infty$. Thus, the result follows. \square

Lemma 4.4. *As $n \rightarrow \infty$,*

$$\sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \left(Z_{k,n}^{5,\ell} - \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R_5^{\theta(\ell), \sigma_0, \lambda_0} \Big| Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right) d\ell \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Proof. First, applying Girsanov's theorem,

$$\begin{aligned}
& \mathbb{E} \left[Z_{k,n}^{5,\ell} - \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R_5^{\theta(\ell), \sigma_0, \lambda_0} \Big| Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \Big| \widehat{\mathcal{F}}_{t_k} \right] \\
& = \Delta_n \sigma_0^{-2} X_{t_k} \mathbb{E}_{\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \left[\left(\widetilde{M}_{t_{k+1}}^{\lambda_0} - \widetilde{M}_{t_k}^{\lambda_0} \right) \frac{d\widehat{P}}{d\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}} \Big| X_{t_k} \right] = 0,
\end{aligned}$$

where we use the independence between $\widetilde{M}_{t_{k+1}}^{\lambda_0} - \widetilde{M}_{t_k}^{\lambda_0}$ and $\frac{d\widehat{P}}{d\widehat{Q}_k^{\theta(\ell), \lambda_0, \theta_0, \lambda_0, \sigma_0}}$. Thus the term appearing in condition (i) of Lemma 3.9 actually equals zero. Next, multiplying the random variable inside the expectation by $\mathbf{1}_{\widehat{J}_{0,k}} + \mathbf{1}_{\widehat{J}_{1,k}} + \mathbf{1}_{\widehat{J}_{\geq 2,k}}$, and applying Lemma 6.6, we get that for n large enough, for any $\alpha \in (0, \frac{1}{2})$, $p_1 > 1$, $q_1 > 1$ with $p_1 q_1 < 2$, and $\mu_1 \in (1, 2)$,

$$\begin{aligned}
& \frac{u^2}{n\Delta_n^3} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\int_0^1 \left(Z_{k,n}^{5,\ell} - \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[R_5^{\theta(\ell), \sigma_0, \lambda_0} \Big| Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right) d\ell \right)^2 \Big| \widehat{\mathcal{F}}_{t_k} \right] \\
& \leq \frac{u^2}{\sigma_0^4 n \Delta_n} \sum_{k=0}^{n-1} X_{t_k}^2 \int_0^1 \mathbb{E} \left[\left(\Delta N_{t_{k+1}} - \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta(\ell), \sigma_0, \lambda_0} \left[\Delta M_{t_{k+1}} \Big| Y_{t_{k+1}}^{\theta(\ell), \sigma_0, \lambda_0} = X_{t_{k+1}} \right] \right)^2 \Big| X_{t_k} \right] d\ell \\
& = \frac{u^2}{\sigma_0^4 n \Delta_n} \sum_{k=0}^{n-1} X_{t_k}^2 \int_0^1 \left(M_{0,1}^{\theta(\ell), \sigma_0, \lambda_0} + M_{1,1}^{\theta(\ell), \sigma_0, \lambda_0} + M_{\geq 2,1}^{\theta(\ell), \sigma_0, \lambda_0} \right) d\ell \\
& \leq \left(\Delta_n^{-1} e^{-C_1 \Delta_n^{2\alpha-1}} + \Delta_n^{\frac{2}{p_1 q_1} - 1} + \Delta_n^{\frac{2}{\mu_1} - 1} \right) \frac{Cu^2}{n} \sum_{k=0}^{n-1} X_{t_k}^2,
\end{aligned}$$

for some constants $C, C_1 > 0$. This converges to zero in $P^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Note that the counting events $\widehat{J}_{0,k}$, $\widehat{J}_{1,k}$, $\widehat{J}_{\geq 2,k}$ and the terms $M_{0,1}^{\theta(\ell), \sigma_0, \lambda_0}$, $M_{1,1}^{\theta(\ell), \sigma_0, \lambda_0}$, $M_{\geq 2,1}^{\theta(\ell), \sigma_0, \lambda_0}$ are defined in Subsections 6.1 and 6.2. Thus, by Lemma 3.9, the desired result follows. \square

Lemma 4.5. As $n \rightarrow \infty$, $\sum_{k=0}^{n-1} T_{k,n} \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0$.

Proof. We write $T_{k,n} = T_{k,n,1} + T_{k,n,2}$, where

$$\begin{aligned} T_{k,n,1} &:= -\frac{uw}{\sigma_0^2 n \Delta_n} \int_0^1 \int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds d\ell, \\ T_{k,n,2} &:= \frac{\theta_n w}{\sigma_0^2 \sqrt{n \Delta_n}} \int_0^1 \left\{ \int_{t_k}^{t_{k+1}} \left(X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k} \right) ds \right. \\ &\quad \left. - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\int_{t_k}^{t_{k+1}} \left(Y_s^{\theta_n, \sigma_0, \lambda(\ell)} - Y_{t_k}^{\theta_n, \sigma_0, \lambda(\ell)} \right) ds \mid Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] \right\} d\ell. \end{aligned}$$

In the same way the terms $M_{k,n,1}$ and $M_{k,n,2}$ of Lemma 4.3 are treated, we obtain that $\sum_{k=0}^{n-1} T_{k,n,1} \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0$, and $\sum_{k=0}^{n-1} T_{k,n,2} \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0$ as $n \rightarrow \infty$. Thus, the result follows. \square

In all what follows, we set $U_k := \Delta N_{t_{k+1}} - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} [\Delta M_{t_{k+1}} \mid Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}}]$.

Lemma 4.6. As $n \rightarrow \infty$, $\sum_{k=0}^{n-1} R_{k,n} \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0$.

Proof. Using $E[\Delta N_{t_{k+1}} \mid X_{t_k}] = \lambda_0 \Delta_n$, $\Delta M_{t_{k+1}} = \tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)} + \lambda(\ell) \Delta_n$, and Lemma 6.7,

$$\begin{aligned} \sum_{k=0}^{n-1} E[R_{k,n} \mid \hat{\mathcal{F}}_{t_k}] &= \frac{w}{\sigma_0^2 \sqrt{n \Delta_n}} \sum_{k=0}^{n-1} \int_0^1 E[U_k \mid X_{t_k}] d\ell \\ &= \frac{w}{\sigma_0^2 \sqrt{n \Delta_n}} \sum_{k=0}^{n-1} \int_0^1 (\lambda_0 \Delta_n - \lambda_0 \Delta_n) d\ell = 0. \end{aligned}$$

Thus the term appearing in condition (i) of Lemma 3.9 actually equals zero.

Next, proceeding as in Lemma 4.4 by multiplying the random variable inside the expectation by $\mathbf{1}_{\hat{\mathcal{J}}_{0,k}} + \mathbf{1}_{\hat{\mathcal{J}}_{1,k}} + \mathbf{1}_{\hat{\mathcal{J}}_{\geq 2,k}}$, and applying Lemma 6.6, we get that for n large enough, for any $\alpha \in (0, \frac{1}{2})$, $p_1 > 1$, $q_1 > 1$ with $p_1 q_1 < 2$, and $\mu_1 \in (1, 2)$,

$$\begin{aligned} \sum_{k=0}^{n-1} E[R_{k,n}^2 \mid \hat{\mathcal{F}}_{t_k}] &\leq \sum_{k=0}^{n-1} \frac{w^2}{\sigma_0^4 n \Delta_n} \int_0^1 E[U_k^2 \mid X_{t_k}] d\ell \\ &= \sum_{k=0}^{n-1} \frac{w^2}{\sigma_0^4 n \Delta_n} \int_0^1 \left(M_{0,1}^{\theta_n, \sigma_0, \lambda(\ell)} + M_{1,1}^{\theta_n, \sigma_0, \lambda(\ell)} + M_{\geq 2,1}^{\theta_n, \sigma_0, \lambda(\ell)} \right) d\ell \\ &\leq C w^2 \left(\Delta_n^{-1} e^{-C_1 \Delta_n^{2\alpha-1}} + \Delta_n^{\frac{2}{p_1 q_1} - 1} + \Delta_n^{\frac{2}{\mu_1} - 1} \right), \end{aligned}$$

for some constants $C, C_1 > 0$, which converges to zero in $P^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Thus, by Lemma 3.9, the result follows. \square

Lemma 4.7. As $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n \Delta_n^2}} \int_0^1 \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[H^{\theta_n, \sigma(\ell), \lambda_n} \mid Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] d\ell \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Proof. For $\mu \in \{1, 2\}$, we set $h_\mu(X_{t_{k+1}}) := \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n}[(H^{\theta_n, \sigma(\ell), \lambda_n})^\mu | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}}]$. First, using Lemma 6.1, (3.14), Lemma 6.5, Jensen's inequality, and (3.15), for n large enough,

$$\begin{aligned}
& \left| \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \mathbb{E} \left[\tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[H^{\theta_n, \sigma(\ell), \lambda_n} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] | \hat{\mathcal{F}}_{t_k} \right] d\ell \right| \\
&= \left| \sum_{k=0}^{n-1} \frac{v e^{w\sqrt{\frac{\Delta_n}{n}}}}{\sqrt{n\Delta_n^2}} \int_0^1 \left\{ \mathbb{E} \left[h_1(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\hat{J}_{0,k}} \left(\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{\theta_n, \sigma(\ell), \lambda_n} - 1 \right) | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{\infty} \frac{\lambda_0^j}{\lambda_n^j} \int_{\Sigma_k^j} \mathbb{E} \left[h_1(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{\theta_n, \sigma(\ell), \lambda_n} - 1 \right) | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] ds_1 \cdots ds_j \right\} d\ell \right| \\
&\leq C \Delta_n^{\frac{1}{2 \max\{\bar{q}_1, \bar{q}_1, q_1\}}} \left(1 + \sum_{j=1}^{\infty} \frac{\lambda_0^j}{\lambda_n^j} \frac{\Delta_n^j}{j!} \left\{ \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\bar{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\bar{p}_1 \bar{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} \right. \right. \\
&\quad \left. \left. + \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\bar{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\bar{p}_1 \bar{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} + \int_0^1 \left(\frac{\lambda_n}{\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}} \right)^{\frac{j}{\bar{q}_2}} \right. \right. \\
&\quad \left. \left. \times \left(e^{-(\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}) \Delta_n} \left(\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}} \right)^j \right)^{\frac{1}{\bar{p}_1 \bar{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} dh \right\} \right) \frac{|v|}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q),
\end{aligned}$$

for some $C, q > 0$, where $(\bar{q}_1, \bar{q}_2, \bar{q}_3)$, $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$, (q_1, q_2, q_3) , $\bar{p}_1, \tilde{p}_1, p_1$ are as in Lemma 6.5. Here $q_{(j)}^{\theta, \sigma, \lambda}(t-s, x, y)$ are defined in Subsection 6.1, $\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{\theta_n, \sigma(\ell), \lambda_n} \equiv \frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta_n, \sigma(\ell), \lambda_n}}(\Delta_n, X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}, X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}, s_1, \dots, s_j)$. Then using the finiteness of the sum w.r.t. j , the right hand side converges to zero in $P^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$.

Next, applying Jensen's and Hölder's inequalities with p_1, p_2 conjugate, together with Lemmas 6.1, 6.2, and (3.15), we get that for n large enough, and some constants $C, q > 0$,

$$\begin{aligned}
& \sum_{k=0}^{n-1} \frac{v^2}{n\Delta_n^2} \mathbb{E} \left[\left(\int_0^1 \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[H^{\theta_n, \sigma(\ell), \lambda_n} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] d\ell \right)^2 | \hat{\mathcal{F}}_{t_k} \right] \\
&\leq \sum_{k=0}^{n-1} \frac{v^2}{n\Delta_n^2} e^{w\sqrt{\frac{\Delta_n}{n}}} \int_0^1 \left\{ \mathbb{E} \left[h_2(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\hat{J}_{0,k}} \frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{\theta_n, \sigma(\ell), \lambda_n} | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] \right. \\
&\quad \left. + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda_n} \right)^j \int_{\Sigma_k^j} \mathbb{E} \left[h_2(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{\theta_n, \sigma(\ell), \lambda_n} | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] ds_1 \cdots ds_j \right\} d\ell \\
&\leq \sum_{k=0}^{n-1} \frac{C v^2}{n\Delta_n^2} \int_0^1 \left(\mathbb{E} \left[|h_2(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n})|^{p_1} | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] \right)^{\frac{1}{p_1}} \\
&\quad \times \left\{ \left(\mathbb{E} \left[\mathbf{1}_{\hat{J}_{0,k}} \left(\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{\theta_n, \sigma(\ell), \lambda_n} \right)^{p_2} | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] \right)^{\frac{1}{p_2}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda_n} \right)^j \int_{\Sigma_k^j} \left(\mathbb{E} \left[\mathbf{1}_{\{\widehat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \right)^{p_2} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] \right)^{\frac{1}{p_2}} ds_1 \cdots ds_j \Big\} d\ell \\
& \leq \sum_{k=0}^{n-1} \frac{Cv^2}{n\Delta_n^2} \int_0^1 (\mathbb{E}[|H^{\theta_n, \sigma(\ell), \lambda_n}|^{2p_1} | Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k}])^{\frac{1}{p_1}} d\ell \left\{ 1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda_n} \right)^j \frac{\Delta_n^j}{j!} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{p_2}} \right\} \\
& \leq C \Delta_n^{\frac{1}{2p_1}} \frac{v^2}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q),
\end{aligned}$$

which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Here we have used the finiteness of the sum w.r.t. j . Thus, by Lemma 3.9, the result desired follows. \square

Lemma 4.8. *As $n \rightarrow \infty$,*

$$\begin{aligned}
& \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \left\{ (H_6 + H_7)^2 + 2\sigma_0 \Delta B_{t_{k+1}} (H_6 + H_7) \right. \\
& \quad \left. - \widetilde{\mathbb{E}}_{X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}} \left[\left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right)^2 + 2\sigma(\ell) \Delta W_{t_{k+1}} \right. \right. \\
& \quad \left. \left. \times \left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right) \middle| Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right\} d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0.
\end{aligned}$$

Proof. We split the term inside the conditional expectation as

$$\left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right)^2 + 2\sigma(\ell) \Delta W_{t_{k+1}} \left(H_4^{\theta_n, \sigma(\ell), \lambda_n} + H_5^{\theta_n, \sigma(\ell), \lambda_n} \right) = K_{n, \ell} + F_{n, \ell},$$

where

$$\begin{aligned}
K_{n, \ell} &:= \theta_n^2 \left(\int_{t_k}^{t_{k+1}} Y_s^{\theta_n, \sigma(\ell), \lambda_n} ds \right)^2 - 2\theta_n \int_{t_k}^{t_{k+1}} Y_s^{\theta_n, \sigma(\ell), \lambda_n} ds \left(\sigma(\ell) \Delta W_{t_{k+1}} + \widetilde{M}_{t_{k+1}}^{\lambda_n} - \widetilde{M}_{t_k}^{\lambda_n} \right), \\
F_{n, \ell} &:= \left(\widetilde{M}_{t_{k+1}}^{\lambda_n} - \widetilde{M}_{t_k}^{\lambda_n} \right)^2 + 2\sigma(\ell) \Delta W_{t_{k+1}} \left(\widetilde{M}_{t_{k+1}}^{\lambda_n} - \widetilde{M}_{t_k}^{\lambda_n} \right).
\end{aligned}$$

Similarly, we split $(H_6 + H_7)^2 + 2\sigma_0 \Delta B_{t_{k+1}} (H_6 + H_7) = K_0 + F_0$, where

$$\begin{aligned}
K_0 &:= \theta_0^2 \left(\int_{t_k}^{t_{k+1}} X_s^{\theta_0, \sigma_0, \lambda_0} ds \right)^2 - 2\theta_0 \int_{t_k}^{t_{k+1}} X_s^{\theta_0, \sigma_0, \lambda_0} ds \left(\sigma_0 \Delta B_{t_{k+1}} + \widetilde{N}_{t_{k+1}}^{\lambda_0} - \widetilde{N}_{t_k}^{\lambda_0} \right), \\
F_0 &:= \left(\widetilde{N}_{t_{k+1}}^{\lambda_0} - \widetilde{N}_{t_k}^{\lambda_0} \right)^2 + 2\sigma_0 \Delta B_{t_{k+1}} \left(\widetilde{N}_{t_{k+1}}^{\lambda_0} - \widetilde{N}_{t_k}^{\lambda_0} \right).
\end{aligned}$$

Therefore, in order to prove Lemma 4.8, it suffices to show that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \left(K_0 - \widetilde{\mathbb{E}}_{X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}} \left[K_{n, \ell} \middle| Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0, \quad (4.1)$$

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \left(F_0 - \widetilde{\mathbb{E}}_{X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}} \left[F_{n, \ell} \middle| Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0. \quad (4.2)$$

We first show (4.1) by proving that conditions (i) and (ii) of Lemma 3.9 hold under $P^{\theta_0, \sigma_0, \lambda_0}$. We set $g(X_{t_{k+1}}) := \widetilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} [K_{n, \ell} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}}]$. Applying Lemma 6.1 to the conditional expectation, we get that

$$E \left[K_0 - \widetilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[K_{n, \ell} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] | \widehat{\mathcal{F}}_{t_k} \right] = S_1 + S_2,$$

where

$$\begin{aligned} S_1 &:= -e^{w\sqrt{\frac{\Delta_n}{n}}} \left\{ E \left[g(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\widehat{J}_{0,k}} \left(\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta_n, \sigma(\ell), \lambda_n}} - 1 \right) | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \frac{\lambda_0^j}{\lambda_n^j} \int_{\Sigma_k^j} E \left[g(X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\{\widehat{J}_{j,k}, s_{1, \dots, s_j}\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} - 1 \right) | X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] ds_1 \cdots ds_j \right\}, \\ S_2 &:= E[K_0 | X_{t_k}] - E[K_{n, \ell} | Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k}]. \end{aligned}$$

Proceeding as in Lemma 4.7, we obtain that

$$\begin{aligned} \left| \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} S_1 d\ell \right| &\leq C \Delta_n^{\frac{1}{2 \max\{\bar{q}_1, \bar{q}_1, q_1\}}} \left(1 + \sum_{j=1}^{\infty} \frac{\lambda_0^j}{\lambda_n^j} \frac{\Delta_n^j}{j!} \left\{ \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\bar{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\bar{p}_1 \bar{q}_2}} \right. \right. \\ &\quad \times (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} + \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\bar{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\bar{p}_1 \bar{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} + \int_0^1 \left(\frac{\lambda_n}{\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}} \right)^{\frac{j}{\bar{q}_2}} \\ &\quad \left. \left. \times \left(e^{-(\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}) \Delta_n} (\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}})^j \right)^{\frac{1}{\bar{p}_1 \bar{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} dh \right\} \right) \frac{|v|}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q), \end{aligned}$$

for some constants $C, q > 0$, where $(\bar{q}_1, \bar{q}_2, \bar{q}_3)$, $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$, (q_1, q_2, q_3) , and $\bar{p}_1, \tilde{p}_1, p_1$ are as in Lemma 6.5. This converges to zero in $P^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. On the other hand,

$$\begin{aligned} E[K_0 | X_{t_k}] &= 2\theta_0^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^s E[X_s^{\theta_0, \sigma_0, \lambda_0} X_u^{\theta_0, \sigma_0, \lambda_0} | X_{t_k}] du ds \\ &\quad - 2\theta_0 \int_{t_k}^{t_{k+1}} E[X_s^{\theta_0, \sigma_0, \lambda_0} (\sigma_0(B_s - B_{t_k}) + \tilde{N}_s^{\lambda_0} - \tilde{N}_{t_k}^{\lambda_0}) | X_{t_k}] ds \\ &= -2\theta_0 \int_{t_k}^{t_{k+1}} E[X_s^{\theta_0, \sigma_0, \lambda_0} (X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k}) | X_{t_k}] ds, \end{aligned}$$

which implies that $S_2 = -2\theta_0 S_{2,1} - S_{2,2}$, where

$$\begin{aligned} S_{2,1} &:= \int_{t_k}^{t_{k+1}} \left(E[X_s^{\theta_0, \sigma_0, \lambda_0} (X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k}) | X_{t_k}] \right. \\ &\quad \left. - E[Y_s^{\theta_n, \sigma(\ell), \lambda_n} (Y_s^{\theta_n, \sigma(\ell), \lambda_n} - Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}) | Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k}] \right) ds, \\ S_{2,2} &:= 2 \frac{u}{\sqrt{n\Delta_n}} \int_{t_k}^{t_{k+1}} E[Y_s^{\theta_n, \sigma(\ell), \lambda_n} (Y_s^{\theta_n, \sigma(\ell), \lambda_n} - Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}) | Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k}] ds. \end{aligned}$$

Setting $h(X_s^{\theta_n, \sigma(\ell), \lambda_n}) := X_s^{\theta_n, \sigma(\ell), \lambda_n} (X_s^{\theta_n, \sigma(\ell), \lambda_n} - X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n})$, $\widehat{J}_{j, t_k, s} := \{N_s - N_{t_k} = j\}$, $\Sigma_{t_k, s}^j := \{(s_1, \dots, s_j) : t_k < s_1 < \dots < s_j < s\}$, and proceeding as in Lemma 6.1, we have

$$S_{2,1} = e^{w\sqrt{\frac{\Delta_n}{n}}} \int_{t_k}^{t_{k+1}} \left\{ \mathbb{E} \left[h(X_s^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\widehat{J}_{0, t_k, s}} \left(\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta_n, \sigma(\ell), \lambda_n}} - 1 \right) \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] + \sum_{j=1}^{\infty} \frac{\lambda_0^j}{\lambda_n^j} \right. \\ \left. \times \int_{\Sigma_{t_k, s}^j} \mathbb{E} \left[h(X_s^{\theta_n, \sigma(\ell), \lambda_n}) \mathbf{1}_{\{\widehat{J}_{j, t_k, s, s_1, \dots, s_j}\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} - 1 \right) \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] ds_1 \cdots ds_j \right\} ds,$$

where we denote $\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta_n, \sigma(\ell), \lambda_n}} \equiv \frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta_n, \sigma(\ell), \lambda_n}}(s - t_k, X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}, X_s^{\theta_n, \sigma(\ell), \lambda_n})$, $\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \equiv \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}}(s - t_k, X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}, X_s^{\theta_n, \sigma(\ell), \lambda_n}; s_1, \dots, s_j)$. Then proceeding as in Lemma 4.7, we obtain that

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} S_{2,1} d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Next, adding and subtracting $Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}$ and use (3.3) to get $S_{2,2} = S_{2,2,1} + S_{2,2,2}$, where

$$S_{2,2,1} := 2 \frac{u}{\sqrt{n\Delta_n}} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\left(Y_s^{\theta_n, \sigma(\ell), \lambda_n} - Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} \right)^2 \middle| Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] ds, \\ S_{2,2,2} := -2\theta_n \frac{u}{\sqrt{n\Delta_n}} X_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^s \mathbb{E} \left[Y_u^{\theta_n, \sigma(\ell), \lambda_n} \middle| Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_k} \right] dud s.$$

Therefore, using Lemma 3.6, we get that

$$\left| \sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} S_{2,2} d\ell \right| \leq C|uv| \frac{\sqrt{\Delta_n}}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q),$$

for some constants $C, q > 0$, which concludes that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} S_{2,2} d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0.$$

This finishes the proof of Lemma 3.9 (i).

Next, proceeding as in the proof of Lemma 4.7 by applying Jensen's and Hölder's inequalities with p_1, p_2 conjugate, and Lemmas 6.1, 6.2, we get that for n large enough,

$$\sum_{k=0}^{n-1} \frac{v^2}{n\Delta_n^2} \mathbb{E} \left[\left(\int_0^1 \frac{1}{\sigma(\ell)^3} \left(K_0 - \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[K_{n, \ell} \middle| Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell \right)^2 \middle| \widehat{\mathcal{F}}_{t_k} \right] \\ \leq \sum_{k=0}^{n-1} \frac{2v^2}{n\Delta_n^2} \int_0^1 \frac{1}{\sigma(\ell)^6} \left\{ \mathbb{E} [K_0^2 | X_{t_k}] + \mathbb{E} \left[\widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[K_{n, \ell}^2 \middle| Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \middle| X_{t_k} \right] \right\} d\ell \\ \leq Cv^2 \sqrt{\Delta_n} + C\Delta_n^{\frac{1}{2p_1}} \frac{v^2}{n} \sum_{k=0}^{n-1} (1 + |X_{t_k}|^q),$$

for some constants $C, q > 0$, which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. This finishes the proof of (4.1).

Finally, it remains to treat (4.2). For this, using equations (3.3) and (1.1) to rewrite the Brownian increments $\Delta W_{t_{k+1}}$, $\Delta B_{t_{k+1}}$ in terms of the process increments $\Delta Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}$ and $\Delta X_{t_{k+1}}^{\theta_0, \sigma_0, \lambda_0}$, Poisson increments, and drift terms, it suffices to show that the following terms converge to zero in $P^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$:

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n}} \int_0^1 \frac{1}{\sigma(\ell)^3} \left(\Delta N_{t_{k+1}} - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\Delta M_{t_{k+1}} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell, \quad (4.3)$$

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n}} \int_0^1 \frac{X_{t_k}}{\sigma(\ell)^3} \left(\Delta N_{t_{k+1}} - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\Delta M_{t_{k+1}} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell, \quad (4.4)$$

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \left((\Delta N_{t_{k+1}})^2 - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[(\Delta M_{t_{k+1}})^2 | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell, \quad (4.5)$$

$$\sum_{k=0}^{n-1} \frac{v}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{\Delta X_{t_{k+1}}}{\sigma(\ell)^3} \left(\Delta N_{t_{k+1}} - \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\Delta M_{t_{k+1}} | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right) d\ell, \quad (4.6)$$

$$\sum_{k=0}^{n-1} \frac{v\theta_0}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{d\ell}{\sigma(\ell)^3} \int_{t_k}^{t_{k+1}} (X_s^{\theta_0, \sigma_0, \lambda_0} - X_{t_k}) ds (\tilde{N}_{t_{k+1}}^{\lambda_0} - \tilde{N}_{t_k}^{\lambda_0}), \quad (4.7)$$

$$\sum_{k=0}^{n-1} \frac{v\theta_n}{\sqrt{n\Delta_n^2}} \int_0^1 \frac{1}{\sigma(\ell)^3} \tilde{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\int_{t_k}^{t_{k+1}} (Y_s^{\theta_n, \sigma(\ell), \lambda_n} - Y_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}) ds \right. \\ \left. \times (\tilde{M}_{t_{k+1}}^{\lambda_n} - \tilde{M}_{t_k}^{\lambda_n}) | Y_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] d\ell. \quad (4.8)$$

We then proceed similarly as in the proof of Lemmas 4.4 and 4.6 using the arguments of large deviation type estimates established in Lemma 6.6 in order to deal with (4.3)-(4.6). Moreover, due to Δ_n appearing in the denominator of (4.5)-(4.6), we need to use in this case the decomposition $\mathbf{1}_{\hat{J}_{0,k}} + \mathbf{1}_{\hat{J}_{1,k}} + \mathbf{1}_{\hat{J}_{2,k}} + \mathbf{1}_{\hat{J}_{\geq 3,k}}$ instead of $\mathbf{1}_{\hat{J}_{0,k}} + \mathbf{1}_{\hat{J}_{1,k}} + \mathbf{1}_{\hat{J}_{\geq 2,k}}$, where $\hat{J}_{\geq 3,k} := \{\Delta N_{t_{k+1}} \geq 3\}$. On the other hand, when treating (4.7)-(4.8), the decreasing rate condition $n\Delta_n^2 \rightarrow 0$ as $n \rightarrow \infty$ is needed for showing Lemma 3.9 (i). Thus, the desired proof is now completed. \square

4.3. Main contributions: LAN property.

Lemma 4.9. *Let $\Gamma(\theta_0, \sigma_0, \lambda_0)$ be defined in Theorem 2.1. Then as $n \rightarrow \infty$,*

$$\sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n} + \beta_{k,n}) \xrightarrow{\mathcal{L}(P^{\theta_0, \sigma_0, \lambda_0})} z^* \mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0)) - \frac{1}{2} z^* \Gamma(\theta_0, \sigma_0, \lambda_0) z.$$

Proof. Applying [9, Lemma 4.3] to $\zeta_{k,n} = \xi_{k,n} + \eta_{k,n} + \beta_{k,n}$, it suffices to show that

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\xi_{k,n} + \eta_{k,n} + \beta_{k,n} | \hat{\mathcal{F}}_{t_k} \right] \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} -\frac{u^2}{2} \frac{1}{2\theta_0} \left(1 + \frac{1}{\sigma_0^2} \right) - \frac{v^2}{2} \frac{2}{\sigma_0^2} - \frac{w^2}{2\sigma_0^2} \left(1 + \frac{\sigma_0^2}{\lambda_0} \right), \quad (4.9)$$

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\xi_{k,n}^2 | \hat{\mathcal{F}}_{t_k} \right] - \left(\mathbb{E} \left[\xi_{k,n} | \hat{\mathcal{F}}_{t_k} \right] \right)^2 \right) \xrightarrow{P^{\theta_0, \sigma_0, \lambda_0}} \frac{u^2}{2\theta_0} \left(1 + \frac{1}{\sigma_0^2} \right), \quad (4.10)$$

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\eta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k} \right] - \left(\mathbb{E} \left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right)^2 \right) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} v^2 \frac{2}{\sigma_0^2}, \quad (4.11)$$

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\beta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k} \right] - \left(\mathbb{E} \left[\beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right)^2 \right) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} \frac{w^2}{\sigma_0^2} \left(1 + \frac{\sigma_0^2}{\lambda_0} \right), \quad (4.12)$$

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\xi_{k,n} \eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] - \mathbb{E} \left[\xi_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \mathbb{E} \left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0, \quad (4.13)$$

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\xi_{k,n} \beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] - \mathbb{E} \left[\xi_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \mathbb{E} \left[\beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0, \quad (4.14)$$

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\eta_{k,n} \beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] - \mathbb{E} \left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \mathbb{E} \left[\beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0, \quad (4.15)$$

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\xi_{k,n}^4 + \eta_{k,n}^4 + \beta_{k,n}^4 | \widehat{\mathcal{F}}_{t_k} \right] \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0. \quad (4.16)$$

Proof of (4.9). Using $\mathbb{E}[\Delta B_{t_{k+1}} | \widehat{\mathcal{F}}_{t_k}] = 0$ and Lemma 3.8, as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\xi_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] = -\frac{u^2}{2\sigma_0^2} \frac{1}{n} \sum_{k=0}^{n-1} X_{t_k}^2 \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} -\frac{u^2}{2\sigma_0^2} \int_{\mathbb{R}} x^2 \pi_{\theta_0, \sigma_0, \lambda_0}(dx).$$

On the other hand, using the ergodicity of $X^{\theta_0, \sigma_0, \lambda_0}$ and Itô's formula, we get that

$$\int_{\mathbb{R}} x^2 \pi_{\theta_0, \sigma_0, \lambda_0}(dx) = \lim_{t \rightarrow +\infty} \mathbb{E} \left[\left(X_t^{\theta_0, \sigma_0, \lambda_0} \right)^2 \right] = \frac{1}{2\theta_0} (\sigma_0^2 + 1). \quad (4.17)$$

Next, since $\mathbb{E}[(\Delta B_{t_{k+1}})^2 | \widehat{\mathcal{F}}_{t_k}] = \Delta_n$, we have that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] = \frac{v}{\sqrt{n}} \sum_{k=0}^{n-1} \int_0^1 \frac{\sigma_0^2 - \sigma(\ell)^2}{\sigma(\ell)^3} d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} -\frac{v^2}{2} \frac{2}{\sigma_0^2}.$$

By Lemma 6.7, we get that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] = -\frac{w^2}{2\sigma_0^2} - \frac{uw}{\sigma_0^2} \frac{1}{n} \sum_{k=0}^{n-1} X_{t_k} - w^2 \int_0^1 \frac{\ell}{\lambda(\ell)} d\ell \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} -\frac{w^2}{2\sigma_0^2} - \frac{w^2}{2\lambda_0}.$$

Here, we have used Lemma 3.8, the ergodicity of $X^{\theta_0, \sigma_0, \lambda_0}$, and (1.2) to get that as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{k=0}^{n-1} X_{t_k} \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} \int_{\mathbb{R}} x \pi_{\theta_0, \sigma_0, \lambda_0}(dx) = \lim_{t \rightarrow +\infty} \mathbb{E} \left[X_t^{\theta_0, \sigma_0, \lambda_0} \right] = 0. \quad (4.18)$$

Thus, we have shown (4.9).

Proof of (4.10). Observe that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\xi_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right)^2 = \frac{u^4}{4\sigma_0^4} \frac{1}{n^2} \sum_{k=0}^{n-1} X_{t_k}^4 \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Using properties of the moments of the Brownian motion, Lemma 3.8 and (4.17), we get

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\xi_{k,n}^2 | \widehat{\mathcal{F}}_{t_k} \right] = \frac{u^2}{\sigma_0^4 n} \sum_{k=0}^{n-1} \left(\sigma_0^2 X_{t_k}^2 + \frac{u^2 X_{t_k}^4}{4n} \right) \xrightarrow{\text{P}^{\theta_0, \sigma_0, \lambda_0}} \frac{u^2}{2\theta_0} \left(1 + \frac{1}{\sigma_0^2} \right),$$

which concludes (4.10).

Proof of (4.11). First, observe that for some constant $C > 0$,

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right)^2 = \frac{v^2}{n} \sum_{k=0}^{n-1} \left(\int_0^1 \frac{\sigma_0^2 - \sigma(\ell)^2}{\sigma(\ell)^3} d\ell \right)^2 \leq \frac{C}{n}.$$

On the other hand, since $\mathbb{E}[(\Delta B_{t_{k+1}})^2 | \widehat{\mathcal{F}}_{t_k}] = \Delta_n$ and $\mathbb{E}[(\Delta B_{t_{k+1}})^4 | \widehat{\mathcal{F}}_{t_k}] = 3\Delta_n^2$, we get

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{E} \left[\eta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k} \right] &= \frac{v^2}{n\Delta_n^2} \sum_{k=0}^{n-1} \left\{ \left(\int_0^1 \frac{\sigma_0^2}{\sigma(\ell)^3} d\ell \right)^2 \mathbb{E} \left[(\Delta B_{t_{k+1}})^4 | \widehat{\mathcal{F}}_{t_k} \right] + \left(\int_0^1 \frac{\Delta_n}{\sigma(\ell)} d\ell \right)^2 \right. \\ &\quad \left. - 2 \int_0^1 \frac{\Delta_n}{\sigma(\ell)} d\ell \int_0^1 \frac{\sigma_0^2}{\sigma(\ell)^3} d\ell \mathbb{E} \left[(\Delta B_{t_{k+1}})^2 | \widehat{\mathcal{F}}_{t_k} \right] \right\} \\ &\rightarrow v^2 \frac{2}{\sigma_0^2}, \end{aligned}$$

as $n \rightarrow \infty$. This concludes (4.11).

Proof of (4.12). First, using (4.17) and Lemma 6.7, we get that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \left(\mathbb{E} \left[\beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right)^2 = \sum_{k=0}^{n-1} \left(-\frac{w^2}{2\sigma_0^2 n} - \frac{uw}{\sigma_0^2 n} X_{t_k} - \frac{w^2}{n} \int_0^1 \frac{\ell}{\lambda(\ell)} d\ell \right)^2 \xrightarrow{\text{P}^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Next, we write $\sum_{k=0}^{n-1} \mathbb{E}[\beta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k}] = S_{n,1} + S_{n,2} - 2S_{n,3}$, where

$$\begin{aligned} S_{n,1} &:= \frac{w^2}{\sigma_0^4 n \Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\sigma_0 \Delta B_{t_{k+1}} + \frac{w \Delta_n}{2\sqrt{n \Delta_n}} + \frac{u \Delta_n}{\sqrt{n \Delta_n}} X_{t_k} \right)^2 | X_{t_k} \right], \\ S_{n,2} &:= \frac{w^2}{n \Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\int_0^1 \frac{1}{\lambda(\ell)} \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell \right)^2 | X_{t_k} \right], \\ S_{n,3} &:= \frac{w^2}{\sigma_0^2 n \Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\sigma_0 \Delta B_{t_{k+1}} + \frac{w \Delta_n}{2\sqrt{n \Delta_n}} + \frac{u \Delta_n}{\sqrt{n \Delta_n}} X_{t_k} \right) \right. \\ &\quad \left. \times \int_0^1 \frac{1}{\lambda(\ell)} \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell | X_{t_k} \right]. \end{aligned}$$

Using properties of the moments of the Brownian motion, (4.17) and (4.18), we get that $S_{n,1} \xrightarrow{\text{P}^{\theta_0, \sigma_0, \lambda_0}} \frac{w^2}{\sigma_0^2}$ as $n \rightarrow \infty$.

Since $\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} = \Delta M_{t_{k+1}} - \lambda(\ell) \Delta_n$, we write $S_{n,2} = S_{n,2,1} - 2S_{n,2,2} + w^2 \Delta_n$, where

$$S_{n,2,1} := \frac{w^2}{n \Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\int_0^1 \frac{1}{\lambda(\ell)} \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\Delta M_{t_{k+1}} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell \right)^2 | X_{t_k} \right],$$

$$S_{n,2,2} := \frac{w^2}{n} \sum_{k=0}^{n-1} \int_0^1 \frac{1}{\lambda(\ell)} \mathbb{E} \left[\tilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\Delta M_{t_{k+1}} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] d\ell.$$

Observe that Lemma 6.7 yields $S_{n,2,2} \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0$. Moreover, adding and subtracting the term $\Delta N_{t_{k+1}}$ inside the integral, we have $S_{n,2,1} = S_{n,2,1,1} + S_{n,2,1,2} - 2S_{n,2,1,3}$, where

$$\begin{aligned} S_{n,2,1,1} &:= \frac{w^2}{n\Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\int_0^1 \frac{1}{\lambda(\ell)} \Delta N_{t_{k+1}} d\ell \right)^2 | X_{t_k} \right], \\ S_{n,2,1,2} &:= \frac{w^2}{n\Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\int_0^1 \frac{1}{\lambda(\ell)} U_k d\ell \right)^2 | X_{t_k} \right], \\ S_{n,2,1,3} &:= \frac{w^2}{n\Delta_n} \sum_{k=0}^{n-1} \mathbb{E} \left[\int_0^1 \frac{1}{\lambda(\ell)} \Delta N_{t_{k+1}} d\ell \int_0^1 \frac{1}{\lambda(\ell)} U_k d\ell | X_{t_k} \right]. \end{aligned}$$

Proceeding as in the proof of Lemma 4.6, one can show that $S_{n,2,1,2}$ and $S_{n,2,1,3}$ converge to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Moreover, since $\mathbb{E}[(\Delta N_{t_{k+1}})^2 | X_{t_k}] = \lambda_0 \Delta_n + (\lambda_0 \Delta_n)^2$, we deduce that $S_{n,2,1,1} \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} \frac{w^2}{\lambda_0}$, which implies that $S_{n,2} \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} \frac{w^2}{\lambda_0}$ as $n \rightarrow \infty$.

Next, we show that $S_{n,3} \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0$ as $n \rightarrow \infty$. Using Lemma 6.7, it suffices to show that

$$S_{n,3,1} = \frac{w^2}{\sigma_0 n \Delta_n} \sum_{k=0}^{n-1} \int_0^1 \frac{1}{\lambda(\ell)} \mathbb{E} \left[\Delta B_{t_{k+1}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] d\ell$$

converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability. For this, using the independence between B and N , and Cauchy-Schwarz inequality, we get that

$$\begin{aligned} |S_{n,3,1}| &= \frac{w^2}{\sigma_0 n \Delta_n} \left| \sum_{k=0}^{n-1} \int_0^1 \frac{1}{\lambda(\ell)} \mathbb{E} [\Delta B_{t_{k+1}} U_k | X_{t_k}] d\ell \right| \\ &\leq \frac{w^2}{\sigma_0 n \sqrt{\Delta_n}} \sum_{k=0}^{n-1} \int_0^1 \frac{1}{|\lambda(\ell)|} (\mathbb{E} [U_k^2 | X_{t_k}])^{1/2} d\ell, \end{aligned}$$

which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$ by proceeding as in Lemma 4.6.

Consequently, the proof of (4.12) is now completed.

Proof of (4.13). Observe that

$$\sum_{k=0}^{n-1} \mathbb{E} [\xi_{k,n} \eta_{k,n} | \widehat{\mathcal{F}}_{t_k}] = \sum_{k=0}^{n-1} \mathbb{E} [\xi_{k,n} | \widehat{\mathcal{F}}_{t_k}] \mathbb{E} [\eta_{k,n} | \widehat{\mathcal{F}}_{t_k}] = \frac{u^2 v^2}{2\sigma_0^2 n^2} \sum_{k=0}^{n-1} X_{t_k}^2 \int_0^1 \frac{\sigma_0 + \sigma(\ell)}{\sigma(\ell)^3} \ell d\ell.$$

Therefore, $\sum_{k=0}^{n-1} (\mathbb{E} [\xi_{k,n} \eta_{k,n} | \widehat{\mathcal{F}}_{t_k}] - \mathbb{E} [\xi_{k,n} | \widehat{\mathcal{F}}_{t_k}] \mathbb{E} [\eta_{k,n} | \widehat{\mathcal{F}}_{t_k}]) = 0$, for all $n \geq 1$.

Proof of (4.14). Using again Lemma 6.7 and Lemma 3.8, we get that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \mathbb{E} [\xi_{k,n} | \widehat{\mathcal{F}}_{t_k}] \mathbb{E} [\beta_{k,n} | \widehat{\mathcal{F}}_{t_k}] = \frac{u^2 w^2}{4\sigma_0^4 n^2} \sum_{k=0}^{n-1} X_{t_k}^2 + \frac{u^3 w}{2\sigma_0^4 n^2} \sum_{k=0}^{n-1} X_{t_k}^3 + \frac{u^2 w^2}{2\sigma_0^2 n} \int_0^1 \frac{\ell}{\lambda(\ell)} d\ell$$

$$\mathbb{P}^{\theta_0, \sigma_0, \lambda_0} \xrightarrow{\quad} 0.$$

Moreover, basic computations yield

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{E} \left[\xi_{k,n} \beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] &= \frac{uw}{\sigma_0^2 n} \sum_{k=0}^{n-1} X_{t_k} + \frac{u^2 w^2}{4\sigma_0^4 n^2} \sum_{k=0}^{n-1} X_{t_k}^2 + \frac{u^3 w}{2\sigma_0^4 n^2} \sum_{k=0}^{n-1} X_{t_k}^3 \\ &- \frac{uw}{\sigma_0 n \Delta_n} \sum_{k=0}^{n-1} X_{t_k} \int_0^1 \frac{1}{\lambda(\ell)} \mathbb{E} \left[\Delta B_{t_{k+1}} \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] d\ell \\ &+ \frac{u^2 w^2}{2\sigma_0^2 n^2} \sum_{k=0}^{n-1} X_{t_k}^2 \int_0^1 \frac{\ell}{\lambda(\ell)} d\ell, \end{aligned}$$

which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Here, we have used Lemma 6.7, Lemma 3.8, (4.17), (4.18), and proceeded as for the term $S_{n,3,1}$.

Proof of (4.15). Using again Lemma 6.7 and (4.18),

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{E} \left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \mathbb{E} \left[\beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] &= \frac{w^2 v^2}{2\sigma_0^2 n} \int_0^1 \frac{\sigma_0 + \sigma(\ell)}{\sigma(\ell)^3} \ell d\ell + \frac{uwv^2}{\sigma_0^2 n^2} \sum_{k=0}^{n-1} X_{t_k} \int_0^1 \frac{\sigma_0 + \sigma(\ell)}{\sigma(\ell)^3} \ell d\ell \\ &+ \frac{w^2 v^2}{n} \int_0^1 \frac{\sigma_0 + \sigma(\ell)}{\sigma(\ell)^3} \ell d\ell \int_0^1 \frac{\ell}{\lambda(\ell)} d\ell, \end{aligned}$$

which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Next,

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{E} \left[\eta_{k,n} \beta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] &= \frac{w^2 v^2}{2\sigma_0^2 n} \int_0^1 \frac{\sigma_0 + \sigma(\ell)}{\sigma(\ell)^3} \ell d\ell + \frac{uwv^2}{\sigma_0^2 n^2} \sum_{k=0}^{n-1} X_{t_k} \int_0^1 \frac{\sigma_0 + \sigma(\ell)}{\sigma(\ell)^3} \ell d\ell \\ &+ \frac{w^2 v}{\sqrt{n}} \int_0^1 \frac{d\ell}{\sigma(\ell)} \int_0^1 \frac{\ell}{\lambda(\ell)} d\ell + \frac{uw\sigma_0^2}{n\Delta_n\sqrt{\Delta_n}} \int_0^1 \frac{d\ell}{\sigma(\ell)^3} \\ &\times \sum_{k=0}^{n-1} \int_0^1 \frac{1}{\lambda(\ell)} \mathbb{E} \left[(\Delta B_{t_{k+1}})^2 \widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] d\ell, \end{aligned}$$

which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$. Here, we have used Lemma 6.7, (4.18), and proceeded as for the term $S_{n,3,1}$.

Proof of (4.16). Using Lemma 3.8, as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\xi_{k,n}^4 | \widehat{\mathcal{F}}_{t_k} \right] \leq \frac{8u^4}{\sigma_0^8 n^2} \sum_{k=0}^{n-1} X_{t_k}^4 \left(3\sigma_0^4 + \frac{u^4 X_{t_k}^4}{16n^2} \right) \xrightarrow{\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}} 0.$$

Next, it is easy to check that for some constant $C > 0$,

$$\sum_{k=0}^{n-1} \mathbb{E} \left[\eta_{k,n}^4 | \widehat{\mathcal{F}}_{t_k} \right] \leq \frac{C}{n}.$$

Finally, applying Jensen's inequality and Lemma 3.8, we get that

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{E} \left[\beta_{k,n}^4 | \widehat{\mathcal{F}}_{t_k} \right] &\leq \frac{8w^4}{n^2 \Delta_n^2 \sigma_0^8} \sum_{k=0}^{n-1} \mathbb{E} \left[\left(\sigma_0 \Delta B_{t_{k+1}} + \frac{w \Delta_n}{2\sqrt{n \Delta_n}} + \frac{u \Delta_n}{\sqrt{n \Delta_n}} X_{t_k} \right)^4 | X_{t_k} \right] \\ &\quad + \frac{8w^4}{n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \frac{1}{\lambda(\ell)^4} \mathbb{E} \left[\widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\left(\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} \right)^4 | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] d\ell, \end{aligned}$$

which converges to zero in $\mathbb{P}^{\theta_0, \sigma_0, \lambda_0}$ -probability as $n \rightarrow \infty$, since $\mathbb{E}[(\Delta B_{t_{k+1}})^4 | X_{t_k}] = 3\Delta_n^2$ and

$$\mathbb{E} \left[\widetilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\left(\widetilde{M}_{t_{k+1}}^{\lambda(\ell)} - \widetilde{M}_{t_k}^{\lambda(\ell)} \right)^4 | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] \leq C \Delta_n,$$

for some constant $C > 0$ and n large enough, using the same arguments as in Lemma 6.7. \square

Consequently, from Lemmas 4.1-4.9, the proof of Theorem 2.1 is now completed.

5. CONCLUSION

Considering the SDEs whose jump part is characterized by the stable laws makes the problem simpler to handle thanks to the semi-explicitness of the density of such processes (see Aït-Sahalia and Jacod [1], and Clément and Gloter [4]). In our context, we have shown that the Malliavin calculus is a powerful tool for the stochastic analysis of the log-likelihood ratio of diffusions with jumps. Besides, we need to condition on the jump structure and use large deviation type results. We believe that the argument we introduced here can be extended to more general cases where the transition density is not explicit, with further arguments. In fact, in [13] we have used the same methodology presented here to treat a multidimensional ergodic diffusion with jumps whose unknown parameter appears only in the drift coefficient. However, there is an extension of the result of this paper that we should think about in our future work. As we mentioned in the Introduction, the case of general SDEs with jumps whose unknown parameters appear in the drift and diffusion coefficients and in the jump component remains an open and difficult problem. This issue will be treated in future research.

6. APPENDIX

6.1. Transition density conditioned on the jump structure. For any $t > s$ and $j \geq 0$, we denote by $q_{(j)}^{\theta, \sigma, \lambda}(t - s, x, y)$ the transition density of $X_t^{\theta, \sigma, \lambda}$ conditioned on $X_s^{\theta, \sigma, \lambda} = x$ and $N_t - N_s = j$. The convolution formula for the sum of independent random variables yields

$$p^{\theta, \sigma, \lambda}(t - s, x, y) = \sum_{j=0}^{\infty} q_{(j)}^{\theta, \sigma, \lambda}(t - s, x, y) e^{-\lambda(t-s)} \frac{(\lambda(t-s))^j}{j!}. \quad (6.1)$$

First, using equation (1.2), we have that

$$\begin{aligned} q_{(0)}^{\theta, \sigma, \lambda}(t - s, x, y) &= \mathbb{P} \left(X_t^{\theta, \sigma, \lambda} = y | X_s^{\theta, \sigma, \lambda} = x, N_t - N_s = 0 \right) \\ &= \mathbb{P} \left(\sigma \int_s^t e^{-\theta(t-u)} dB_u = y - x e^{-\theta(t-s)} + \frac{\lambda}{\theta} (1 - e^{-\theta(t-s)}) | X_s^{\theta, \sigma, \lambda} = x, N_t - N_s = 0 \right) \end{aligned}$$

$$= \frac{1}{\sqrt{\frac{\pi}{\theta}\sigma^2(1 - e^{-2\theta(t-s)})}} \exp \left\{ -\frac{(y - xe^{-\theta(t-s)} + \frac{\lambda}{\theta}(1 - e^{-\theta(t-s)}))^2}{\frac{1}{\theta}\sigma^2(1 - e^{-2\theta(t-s)})} \right\}. \quad (6.2)$$

For any $j \geq 1$ and $s < s_1 < \dots < s_j < t$, we denote by $q_{(j)}^{\theta,\sigma,\lambda}(t-s, x, y; s_1, \dots, s_j)$ the transition density of $X_t^{\theta,\sigma,\lambda}$ conditioned on $X_s^{\theta,\sigma,\lambda} = x$, $N_t - N_s = j$ and $(T_1^{s,t} = s_1, \dots, T_j^{s,t} = s_j)$, where $T_1^{s,t}, T_2^{s,t}, \dots$ with $s < T_1^{s,t} < T_2^{s,t} < \dots < t$ are the jump times on $[s, t]$ of the Poisson process N . Given $N_t - N_s = j$, by conditioning on the jump times $(T_1^{s,t}, \dots, T_j^{s,t})$ which are then distributed as the order statistics of j independent uniform random variables on $[s, t]$, we have

$$\begin{aligned} q_{(j)}^{\theta,\sigma,\lambda}(t-s, x, y) &= \mathbb{P} \left(X_t^{\theta,\sigma,\lambda} = y | X_s^{\theta,\sigma,\lambda} = x, N_t - N_s = j \right) \\ &= \int_{\{s < s_1 < \dots < s_j < t\}} \mathbb{P} \left(X_t^{\theta,\sigma,\lambda} = y | X_s^{\theta,\sigma,\lambda} = x, N_t - N_s = j, T_1^{s,t} = s_1, \dots, T_j^{s,t} = s_j \right) \\ &\quad \times \mathbb{P} \left(T_1^{s,t} \in ds_1, \dots, T_j^{s,t} \in ds_j | X_s^{\theta,\sigma,\lambda} = x, N_t - N_s = j \right) \\ &= \frac{j!}{(t-s)^j} \int_{\{s < s_1 < \dots < s_j < t\}} q_{(j)}^{\theta,\sigma,\lambda}(t-s, x, y; s_1, \dots, s_j) ds_1 \dots ds_j. \end{aligned} \quad (6.3)$$

Moreover, using again equation (1.2), definition of stochastic integral w.r.t. Poisson process and setting $\Sigma := \frac{1}{\theta}\sigma^2(1 - e^{-2\theta(t-s)})$, we get that

$$\begin{aligned} q_{(j)}^{\theta,\sigma,\lambda}(t-s, x, y; s_1, \dots, s_j) &= \mathbb{P} \left(X_t^{\theta,\sigma,\lambda} = y | X_s^{\theta,\sigma,\lambda} = x, N_t - N_s = j, T_1^{s,t} = s_1, \dots, T_j^{s,t} = s_j \right) \\ &= \mathbb{P} \left(\sigma \int_s^t e^{-\theta(t-u)} dB_u + \int_s^t e^{-\theta(t-u)} dN_u = y - xe^{-\theta(t-s)} \right. \\ &\quad \left. + \frac{\lambda}{\theta}(1 - e^{-\theta(t-s)}) | X_s^{\theta,\sigma,\lambda} = x, N_t - N_s = j, T_1^{s,t} = s_1, \dots, T_j^{s,t} = s_j \right) \\ &= \mathbb{P} \left(\sigma \int_s^t e^{-\theta(t-u)} dB_u = y - xe^{-\theta(t-s)} + \frac{\lambda}{\theta}(1 - e^{-\theta(t-s)}) \right. \\ &\quad \left. - \left(e^{-\theta(t-s_1)} + \dots + e^{-\theta(t-s_j)} \right) | X_s^{\theta,\sigma,\lambda} = x, N_t - N_s = j, T_1^{s,t} = s_1, \dots, T_j^{s,t} = s_j \right) \\ &= \frac{1}{\sqrt{\pi\Sigma}} \exp \left\{ -\frac{(y - xe^{-\theta(t-s)} + \frac{\lambda}{\theta}(1 - e^{-\theta(t-s)}) - (e^{-\theta(t-s_1)} + \dots + e^{-\theta(t-s_j)}))^2}{\Sigma} \right\}. \end{aligned} \quad (6.4)$$

For $k \in \{0, \dots, n-1\}$, consider the events $\widehat{J}_{j,k} := \{\Delta N_{t_{k+1}} = j\}$ and $\widetilde{J}_{j,k} := \{\Delta M_{t_{k+1}} = j\}$. By conditioning on all the possible number of jumps and jump times of the Poisson process occurring on $[t_k, t_{k+1}]$, we have the following change of measures via the transition density conditioned on the number of jumps and jump times.

Lemma 6.1. *Let f be any bounded function. For any $k \in \{0, \dots, n-1\}$, $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$,*

$$\mathbb{E} [f(X_{t_{k+1}}) | X_{t_k}] = \mathbb{E} \left[f(X_{t_{k+1}}^{\theta,\sigma,\lambda}) D(\Delta_n, X_{t_k}^{\theta,\sigma,\lambda}, X_{t_{k+1}}^{\theta,\sigma,\lambda}) | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right],$$

where, setting $\Sigma_k^j := \{(s_1, \dots, s_j) : t_k < s_1 < \dots < s_j < t_{k+1}\}$,

$$D(\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}) := e^{(\lambda - \lambda_0)\Delta_n} \left(\mathbf{1}_{\widehat{J}_{0,k}} \frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta, \sigma, \lambda}} (\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}) \right. \\ \left. + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda} \right)^j \int_{\Sigma_k^j} \mathbf{1}_{\{\widehat{J}_{j,k}, s_1, \dots, s_j\}} \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta, \sigma, \lambda}} (\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}; s_1, \dots, s_j) ds_1 \cdots ds_j \right),$$

and $\{\widehat{J}_{j,k}, s_1, \dots, s_j\} := \{\Delta N_{t_{k+1}} = j, T_1^{t_k, t_{k+1}} = s_1, \dots, T_j^{t_k, t_{k+1}} = s_j\}$.

Proof. Observe that

$$\begin{aligned} \mathbb{E}[f(X_{t_{k+1}})|X_{t_k}] &= \mathbb{E}[\mathbf{1}_{\widehat{J}_{0,k}} f(X_{t_{k+1}})|X_{t_k}] + \sum_{j=1}^{\infty} \int_{\Sigma_k^j} \mathbb{E}[\mathbf{1}_{\{\widehat{J}_{j,k}, s_1, \dots, s_j\}} f(X_{t_{k+1}})|X_{t_k}] ds_1 \cdots ds_j \\ &= \int_{\mathbb{R}} f(y) q_{(0)}^{\theta_0, \sigma_0, \lambda_0}(\Delta_n, X_{t_k}, y) dy e^{-\lambda_0 \Delta_n} \\ &\quad + \sum_{j=1}^{\infty} \int_{\Sigma_k^j} \int_{\mathbb{R}} f(y) q_{(j)}^{\theta_0, \sigma_0, \lambda_0}(\Delta_n, X_{t_k}, y; s_1, \dots, s_j) dy e^{-\lambda_0 \Delta_n} \frac{(\lambda_0 \Delta_n)^j}{j!} \frac{j!}{\Delta_n^j} ds_1 \cdots ds_j \\ &= e^{(\lambda - \lambda_0)\Delta_n} \left(\int_{\mathbb{R}} f(y) \frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta, \sigma, \lambda}} q_{(j)}^{\theta, \sigma, \lambda}(\Delta_n, X_{t_k}, y) dy e^{-\lambda \Delta_n} \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda} \right)^j \int_{\Sigma_k^j} \int_{\mathbb{R}} f(y) \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta, \sigma, \lambda}} q_{(j)}^{\theta, \sigma, \lambda}(\Delta_n, X_{t_k}, y; s_1, \dots, s_j) dy e^{-\lambda \Delta_n} \frac{(\lambda \Delta_n)^j}{j!} \frac{j!}{\Delta_n^j} ds_1 \cdots ds_j \right), \end{aligned}$$

which gives the desired result. \square

Next, we have the first crucial estimate.

Lemma 6.2. *Let $(\theta, \sigma, \lambda), (\bar{\theta}, \bar{\sigma}, \bar{\lambda}) \in \Theta \times \Sigma \times \Lambda$, and set $\sigma_1 := \frac{\bar{\sigma}^2}{\bar{\theta}}(1 - e^{-2\bar{\theta}\Delta_n})$ and $\sigma_2 := \frac{\sigma^2}{\theta}(1 - e^{-2\theta\Delta_n})$. Then there exists a constant $C > 0$ such that for all $k \in \{0, \dots, n-1\}$ and $j \geq 1$, the following estimates*

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{\widehat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}}{q_{(j)}^{\theta, \sigma, \lambda}} (\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}; s_1, \dots, s_j) \right)^p \middle| X_{t_k}^{\theta, \sigma, \lambda} = x \right] &\leq C e^{-\lambda \Delta_n} \lambda^j, \\ \mathbb{E} \left[\mathbf{1}_{\widehat{J}_{0,k}} \left(\frac{q_{(0)}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}}{q_{(0)}^{\theta, \sigma, \lambda}} (\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}) \right)^p \middle| X_{t_k}^{\theta, \sigma, \lambda} = x \right] &\leq C, \end{aligned}$$

hold for any $p > 1$ if $\sigma_2 \geq \sigma_1$, and for any $p \in (1, \frac{\sigma_1}{\sigma_1 - \sigma_2})$ if $\sigma_2 < \sigma_1$. Moreover, this statement remains valid for $Y^{\theta, \sigma, \lambda}$.

Proof. Using (6.4), we get that for any $p > 1$ if $\sigma_2 \geq \sigma_1$, and $p \in (1, \frac{\sigma_1}{\sigma_1 - \sigma_2})$ if $\sigma_2 < \sigma_1$,

$$\mathbb{E} \left[\mathbf{1}_{\{\widehat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}}{q_{(j)}^{\theta, \sigma, \lambda}} (\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}; s_1, \dots, s_j) \right)^p \middle| X_{t_k}^{\theta, \sigma, \lambda} = x \right]$$

$$\begin{aligned}
&= e^{-\lambda \Delta_n} \frac{(\lambda \Delta_n)^j}{j!} \frac{j!}{\Delta_n^j} \int_{\mathbb{R}} \bar{q}_{(j)}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}(\Delta_n, x, y; s_1, \dots, s_j)^p q_{(j)}^{\theta, \sigma, \lambda}(\Delta_n, x, y; s_1, \dots, s_j)^{1-p} dy \\
&= e^{-\lambda \Delta_n} \lambda^j \int_{\mathbb{R}} \frac{1}{(\pi \sigma_1)^{\frac{p}{2}}} \frac{1}{(\pi \sigma_2)^{\frac{1-p}{2}}} \\
&\quad \times \exp \left\{ -p \frac{\left(y - x e^{-\bar{\theta} \Delta_n} + \frac{\bar{\lambda}}{\bar{\theta}} (1 - e^{-\bar{\theta} \Delta_n}) - (e^{-\bar{\theta}(t_{k+1}-s_1)} + \dots + e^{-\bar{\theta}(t_{k+1}-s_j)}) \right)^2}{\sigma_1} \right\} \\
&\quad \times \exp \left\{ -(1-p) \frac{\left(y - x e^{-\theta \Delta_n} + \frac{\lambda}{\theta} (1 - e^{-\theta \Delta_n}) - (e^{-\theta(t_{k+1}-s_1)} + \dots + e^{-\theta(t_{k+1}-s_j)}) \right)^2}{\sigma_2} \right\} dy \\
&\leq C e^{-\lambda \Delta_n} \lambda^j,
\end{aligned}$$

for some constant $C > 0$, where we use the fact that the dy integral is finite since $\frac{p}{\sigma_1} + \frac{1-p}{\sigma_2} > 0$ for any $p > 1$ if $\sigma_2 \geq \sigma_1$, and $p \in (1, \frac{\sigma_1}{\sigma_1 - \sigma_2})$ if $\sigma_2 < \sigma_1$. This shows the first inequality. Using (6.2) and the same arguments as above, we conclude the second inequality. \square

As in Propositions 3.1 and 3.3, we have the following explicit expression for the logarithm derivatives of the transition density conditioned on the number of jumps and jump times.

Lemma 6.3. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $k \in \{0, \dots, n-1\}$, $j \geq 1$, $\beta \in \{\theta, \sigma\}$, and $x, y \in \mathbb{R}$,*

$$\begin{aligned}
\frac{\partial_{\beta} q_{(j)}^{\theta, \sigma, \lambda}}{q_{(j)}^{\theta, \sigma, \lambda}}(\Delta_n, x, y; s_1, \dots, s_j) &= \frac{1}{\Delta_n} \tilde{\mathbb{E}}_x^{\theta, \sigma, \lambda} \left[\delta \left(\partial_{\beta} Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U_t^{\theta, \sigma, \lambda}(t_k, x) \right) \middle| Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y, \tilde{J}_{j,k}, s_1, \dots, s_j \right], \\
\frac{\partial_{\lambda} q_{(j)}^{\theta, \sigma, \lambda}}{q_{(j)}^{\theta, \sigma, \lambda}}(\Delta_n, x, y; s_1, \dots, s_j) &= \tilde{\mathbb{E}}_x^{\theta, \sigma, \lambda} \left[-\frac{\Delta W_{t_{k+1}}}{\sigma} + \frac{\tilde{M}_{t_{k+1}}^{\lambda} - \tilde{M}_{t_k}^{\lambda}}{\lambda} \middle| Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y, \tilde{J}_{j,k}, s_1, \dots, s_j \right], \\
\frac{\partial_{\beta} q_{(0)}^{\theta, \sigma, \lambda}}{q_{(0)}^{\theta, \sigma, \lambda}}(\Delta_n, x, y) &= \frac{1}{\Delta_n} \tilde{\mathbb{E}}_x^{\theta, \sigma, \lambda} \left[\delta \left(\partial_{\beta} Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U_t^{\theta, \sigma, \lambda}(t_k, x) \right) \middle| Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y, \tilde{J}_{0,k} \right], \\
\frac{\partial_{\lambda} q_{(0)}^{\theta, \sigma, \lambda}}{q_{(0)}^{\theta, \sigma, \lambda}}(\Delta_n, x, y) &= \tilde{\mathbb{E}}_x^{\theta, \sigma, \lambda} \left[-\frac{\Delta W_{t_{k+1}}}{\sigma} + \frac{\tilde{M}_{t_{k+1}}^{\lambda} - \tilde{M}_{t_k}^{\lambda}}{\lambda} \middle| Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y, \tilde{J}_{0,k} \right],
\end{aligned}$$

where the process $U_t^{\theta, \sigma, \lambda}(t_k, x) = (U_t^{\theta, \sigma, \lambda}(t_k, x), t \in [t_k, t_{k+1}])$ is defined in Proposition 3.1.

Proof. Let f be a continuously differentiable function with compact support. The chain rule of the Malliavin calculus implies that $f'(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) = D_t(f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x))) U_t^{\theta, \sigma, \lambda}(t_k, x)$, for all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ and $t \in [t_k, t_{k+1}]$, where $U_t^{\theta, \sigma, \lambda}(t_k, x) := (D_t Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x))^{-1}$.

Using the Malliavin calculus integration by parts formula on $[t_k, t_{k+1}]$, and the independence between W and $(M, T_1^{t_k, t_{k+1}}, \dots, T_j^{t_k, t_{k+1}})$, we get that

$$\begin{aligned}
\partial_{\beta} \tilde{\mathbb{E}} \left[\mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \right] &= \tilde{\mathbb{E}} \left[\mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} f'(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \partial_{\beta} Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) \right] \\
&= \frac{1}{\Delta_n} \tilde{\mathbb{E}} \left[\int_{t_k}^{t_{k+1}} D_t(f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x))) U_t^{\theta, \sigma, \lambda}(t_k, x) \partial_{\beta} Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) \mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} dt \right]
\end{aligned}$$

$$= \frac{1}{\Delta_n} \tilde{\mathbb{E}} \left[\mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \delta \left(\partial_\beta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) \right].$$

On the other hand, using the stochastic flow property $Y_t^{\theta, \sigma, \lambda} = Y_t^{\theta, \sigma, \lambda}(s, Y_s^{\theta, \sigma, \lambda})$ for all $0 \leq s \leq t$, and the Markov property of diffusion processes, we have that

$$\partial_\beta \tilde{\mathbb{E}} \left[\mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \right] = \int_{\mathbb{R}} f(y) \partial_\beta q_{(j)}^{\theta, \sigma, \lambda}(\Delta_n, x, y; s_1, \dots, s_j) e^{-\lambda \Delta_n} \frac{(\lambda \Delta_n)^j}{j!} \frac{j!}{\Delta_n^j} dy,$$

and

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} f(Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x)) \delta \left(\partial_\beta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) \right] \\ &= \int_{\mathbb{R}} f(y) \tilde{\mathbb{E}} \left[\delta \left(\partial_\beta Y_{t_{k+1}}^{\theta, \sigma, \lambda}(t_k, x) U^{\theta, \sigma, \lambda}(t_k, x) \right) \mid Y_{t_{k+1}}^{\theta, \sigma, \lambda} = y, Y_{t_k}^{\theta, \sigma, \lambda} = x, \tilde{J}_{j,k}, s_1, \dots, s_j \right] \\ & \quad \times q_{(j)}^{\theta, \sigma, \lambda}(\Delta_n, x, y; s_1, \dots, s_j) e^{-\lambda \Delta_n} \frac{(\lambda \Delta_n)^j}{j!} \frac{j!}{\Delta_n^j} dy. \end{aligned}$$

This shows the first equality. Moreover, using the same above arguments and proceeding similarly as in the proof of Proposition 3.3, we derive the other expressions. \square

As a consequence, we have the following estimates.

Lemma 6.4. *Let $(\theta, \sigma, \lambda), (\bar{\theta}, \bar{\sigma}, \bar{\lambda}) \in \Theta \times \Sigma \times \Lambda$, and set $\sigma_1 := \frac{\bar{\sigma}^2}{\bar{\theta}}(1 - e^{-2\bar{\theta}\Delta_n})$ and $\sigma_2 := \frac{\sigma^2}{\theta}(1 - e^{-2\theta\Delta_n})$. Then for any $p > 1$, p_1 and p_2 conjugate with $p_1 > 1$ if $\sigma_2 \geq \sigma_1$, and $p_1 \in (1, \frac{\sigma_1}{\sigma_1 - \sigma_2})$ if $\sigma_2 < \sigma_1$, there exist constants $C, q > 0$ such that for all $k \in \{0, \dots, n-1\}$ and $j \geq 1$,*

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{\partial_\theta q_{(j)}^{\theta, \sigma, \lambda}}{q_{(j)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}; s_1, \dots, s_j) \right)^p \mid X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = x \right] \\ & \leq C \Delta_n^{\frac{p}{2}} e^{(\lambda - \bar{\lambda})\Delta_n} \left(\frac{\bar{\lambda}}{\lambda} \right)^j (e^{-\lambda \Delta_n} \lambda^j)^{\frac{1}{p_1}} (1 + |x|^q), \\ & \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{\partial_\sigma q_{(j)}^{\theta, \sigma, \lambda}}{q_{(j)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}; s_1, \dots, s_j) \right)^p \mid X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = x \right] \\ & \leq C e^{(\lambda - \bar{\lambda})\Delta_n} \left(\frac{\bar{\lambda}}{\lambda} \right)^j (e^{-\lambda \Delta_n} \lambda^j)^{\frac{1}{p_1}} (1 + |x|^q), \\ & \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{\partial_\lambda q_{(j)}^{\theta, \sigma, \lambda}}{q_{(j)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}; s_1, \dots, s_j) \right)^p \mid X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = x \right] \\ & \leq C \Delta_n^{\frac{1}{p_2}} e^{(\lambda - \bar{\lambda})\Delta_n} \left(\frac{\bar{\lambda}}{\lambda} \right)^j (e^{-\lambda \Delta_n} \lambda^j)^{\frac{1}{p_1}}, \\ & \mathbb{E} \left[\mathbf{1}_{\hat{J}_{0,k}} \left(\frac{\partial_\theta q_{(0)}^{\theta, \sigma, \lambda}}{q_{(0)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}}) \right)^p \mid X_{t_k}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = x \right] \leq C \Delta_n^{\frac{p}{2}} e^{(\lambda - \bar{\lambda})\Delta_n} (1 + |x|^q), \end{aligned}$$

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\hat{J}_{0,k}} \left(\frac{\partial_\sigma q_{(0)}^{\theta,\sigma,\lambda}}{q_{(0)}^{\theta,\sigma,\lambda}} (\Delta_n, X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}) \right)^p \mid X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = x \right] &\leq C e^{(\lambda-\bar{\lambda})\Delta_n} (1 + |x|^q), \\ \mathbb{E} \left[\mathbf{1}_{\hat{J}_{0,k}} \left(\frac{\partial_\lambda q_{(0)}^{\theta,\sigma,\lambda}}{q_{(0)}^{\theta,\sigma,\lambda}} (\Delta_n, X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}) \right)^p \mid X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = x \right] &\leq C \Delta_n^{\frac{1}{p_2}} e^{(\lambda-\bar{\lambda})\Delta_n}. \end{aligned}$$

Proof. Applying Lemma 6.3 and Jensen's inequality, we obtain that for any $j \geq 1$ and $p > 1$,

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{\partial_\sigma q_{(j)}^{\theta,\sigma,\lambda}}{q_{(j)}^{\theta,\sigma,\lambda}} (\Delta_n, X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}; s_1, \dots, s_j) \right)^p \mid X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = x \right] &= \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \right. \\ &\times \left. \left(\frac{1}{\Delta_n} \tilde{\mathbb{E}}_x^{\theta,\sigma,\lambda} \left[\delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) \mid Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}, \tilde{J}_{j,k}, s_1, \dots, s_j \right] \right)^p \mid X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = x \right] \\ &\leq \frac{1}{\Delta_n^p} \int_{\mathbb{R}} \tilde{\mathbb{E}}_x^{\theta,\sigma,\lambda} \left[\left| \delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) \right|^p \mid Y_{t_{k+1}}^{\theta,\sigma,\lambda} = y, \tilde{J}_{j,k}, s_1, \dots, s_j \right] \\ &\quad \times q_{(j)}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}(\Delta_n, x, y; s_1, \dots, s_j) e^{-\bar{\lambda}\Delta_n} \frac{(\bar{\lambda}\Delta_n)^j}{j!} \frac{j!}{\Delta_n^j} dy \\ &= \frac{1}{\Delta_n^p} e^{(\lambda-\bar{\lambda})\Delta_n} \left(\frac{\bar{\lambda}}{\lambda} \right)^j \\ &\quad \times \tilde{\mathbb{E}}_x^{\theta,\sigma,\lambda} \left[\left| \delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) \right|^p \mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} \frac{q_{(j)}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}}{q_{(j)}^{\theta,\sigma,\lambda}} (\Delta_n, Y_{t_k}^{\theta,\sigma,\lambda}, Y_{t_{k+1}}^{\theta,\sigma,\lambda}; s_1, \dots, s_j) \right]. \end{aligned}$$

Next applying Hölder's inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, Lemma 6.2 and (3.16), we get that for any $j \geq 1$, $p > 1$, $p_1 > 1$ if $\sigma_2 \geq \sigma_1$, and $p_1 \in (1, \frac{\sigma_1}{\sigma_1 - \sigma_2})$ if $\sigma_2 < \sigma_1$,

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{\partial_\sigma q_{(j)}^{\theta,\sigma,\lambda}}{q_{(j)}^{\theta,\sigma,\lambda}} (\Delta_n, X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}, X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}; s_1, \dots, s_j) \right)^p \mid X_{t_k}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = x \right] \\ \leq \frac{1}{\Delta_n^p} e^{(\lambda-\bar{\lambda})\Delta_n} \left(\frac{\bar{\lambda}}{\lambda} \right)^j \left(\tilde{\mathbb{E}}_x^{\theta,\sigma,\lambda} \left[\left| \delta \left(\partial_\sigma Y_{t_{k+1}}^{\theta,\sigma,\lambda}(t_k, x) U^{\theta,\sigma,\lambda}(t_k, x) \right) \right|^{pp_2} \right] \right)^{\frac{1}{p_2}} \\ \times \left(\tilde{\mathbb{E}}_x^{\theta,\sigma,\lambda} \left[\mathbf{1}_{\{\tilde{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}}{q_{(j)}^{\theta,\sigma,\lambda}} (\Delta_n, Y_{t_k}^{\theta,\sigma,\lambda}, Y_{t_{k+1}}^{\theta,\sigma,\lambda}; s_1, \dots, s_j) \right)^{p_1} \right] \right)^{\frac{1}{p_1}} \\ \leq C e^{(\lambda-\bar{\lambda})\Delta_n} \left(\frac{\bar{\lambda}}{\lambda} \right)^j (e^{-\lambda\Delta_n} \lambda^j)^{\frac{1}{p_1}} (1 + |x|^q), \end{aligned}$$

for some constants $C, q > 0$. This concludes the second inequality. The proof of the others follows along the same lines and is thus omitted. \square

As a consequence, we have the following crucial estimate.

Lemma 6.5. *For any random variable Z , $k \in \{0, \dots, n-1\}$ and $j \geq 1$, there exist constants $C, q > 0$ such that for n large enough, for any $\bar{q}_1, \bar{q}_2, \bar{q}_3 > 1$ conjugate, $\bar{p}_1 > 1$, $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3 > 1$*

conjugate, $\tilde{p}_1 > 1$, $q_1, q_2, q_3 > 1$ conjugate, and $p_1 > 1$,

$$\begin{aligned} & \left| \mathbb{E} \left[Z \mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} (\Delta_n, X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}, X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}; s_1, \dots, s_j) - 1 \right) \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right| \\ & \leq \frac{C}{\sqrt{n}} (1 + |x|^q) \left\{ \left(\mathbb{E} \left[|Z|^{\tilde{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\tilde{q}_1}} \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\tilde{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\tilde{p}_1 \tilde{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\tilde{q}_3}} \right. \\ & \quad + \left(\mathbb{E} \left[|Z|^{\tilde{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\tilde{q}_1}} \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\tilde{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\tilde{p}_1 \tilde{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\tilde{q}_3}} + \int_0^1 \left(\frac{\lambda_n}{\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}} \right)^{\frac{j}{\tilde{q}_2}} \\ & \quad \times \left. \left(\mathbb{E} \left[|Z|^{q_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{q_1}} \left(e^{-(\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}) \Delta_n} (\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}})^j \right)^{\frac{1}{p_1 q_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{q_3}} dh \right\}, \end{aligned}$$

and

$$\begin{aligned} & \left| \mathbb{E} \left[Z \mathbf{1}_{\hat{J}_{0,k}} \left(\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta_n, \sigma(\ell), \lambda_n}} (\Delta_n, X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}, X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}) - 1 \right) \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right| \\ & \leq \frac{C}{\sqrt{n}} (1 + |x|^q) \left\{ \left(\mathbb{E} \left[|Z|^{\tilde{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\tilde{q}_1}} + \left(\mathbb{E} \left[|Z|^{\tilde{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\tilde{q}_1}} \right. \\ & \quad \left. + \left(\mathbb{E} \left[|Z|^{q_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{q_1}} \right\}. \end{aligned}$$

Proof. We write $\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \equiv \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} (\Delta_n, X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n}, X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n}; s_1, \dots, s_j)$. Observe that

$$\begin{aligned} & \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} - 1 = \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0} - q_{(j)}^{\theta_0, \sigma(\ell), \lambda_0} + q_{(j)}^{\theta_0, \sigma(\ell), \lambda_0} - q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0} + q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0} - q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \\ & = - \int_0^1 \left(\frac{\ell v}{\sqrt{n}} \frac{\partial_{\sigma} q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell v h}{\sqrt{n}}, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} + \frac{u}{\sqrt{n\Delta_n}} \frac{\partial_{\theta} q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} + \frac{w}{\sqrt{n\Delta_n}} \frac{\partial_{\lambda} q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \right) dh. \end{aligned}$$

This implies that

$$\left| \mathbb{E} \left[Z \mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} - 1 \right) \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right| \leq S_1 + S_2 + S_3,$$

where

$$\begin{aligned} S_1 &:= \frac{|v|}{\sqrt{n}} \int_0^1 \left| \mathbb{E} \left[Z \mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \frac{\partial_{\sigma} q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell v h}{\sqrt{n}}, \lambda_0}}{q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell v h}{\sqrt{n}}, \lambda_0}} \frac{q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell v h}{\sqrt{n}}, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right| dh, \\ S_2 &:= \frac{|u|}{\sqrt{n\Delta_n}} \int_0^1 \left| \mathbb{E} \left[Z \mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \frac{\partial_{\theta} q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}}{q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}} \frac{q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right| dh, \end{aligned}$$

$$S_3 := \frac{|w|}{\sqrt{n\Delta_n}} \int_0^1 \left| \mathbb{E} \left[Z \mathbf{1}_{\{\hat{J}_{j,k,s_1}, \dots, s_j\}} \frac{\partial_{\lambda} q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}}}{\frac{\partial_{\lambda} q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}}}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right| dh.$$

First, using Hölder's inequality with $\bar{q}_1, \bar{q}_2, \bar{q}_3 > 1$ conjugate, Lemmas 6.2 and 6.4 where notice that θ_n tends to θ_0 , and $\sigma_0 + \frac{\ell vh}{\sqrt{n}}$ and $\sigma(\ell)$ tend to σ_0 as $n \rightarrow \infty$, we get that for n large enough, and for any $\bar{p}_1 > 1$,

$$\begin{aligned} S_1 &\leq \frac{|v|}{\sqrt{n}} \int_0^1 \left(\mathbb{E} \left[|Z|^{\bar{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x| \right] \right)^{\frac{1}{\bar{q}_1}} \\ &\quad \times \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k,s_1}, \dots, s_j\}} \left(\frac{\partial_{\sigma} q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell vh}{\sqrt{n}}, \lambda_0}}{q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell vh}{\sqrt{n}}, \lambda_0}} \right)^{\bar{q}_2} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\bar{q}_2}} \\ &\quad \times \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k,s_1}, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0 + \frac{\ell vh}{\sqrt{n}}, \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \right)^{\bar{q}_3} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\bar{q}_3}} dh \\ &\leq C \frac{|v|}{\sqrt{n}} (\mathbb{E}[|Z|^{\bar{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x])^{\frac{1}{\bar{q}_1}} e^{-\frac{w}{\bar{q}_2} \sqrt{\frac{\Delta_n}{n}}} \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\bar{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\bar{p}_1 \bar{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\bar{q}_3}} (1 + |x|^q). \end{aligned}$$

Next, using Hölder's inequality with $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3 > 1$ conjugate, Lemmas 6.2 and 6.4 where notice that $\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}$ and θ_n tend to θ_0 as $n \rightarrow \infty$, we get that for n large enough, and any $\tilde{p}_1 > 1$,

$$\begin{aligned} S_2 &\leq \frac{|u|}{\sqrt{n\Delta_n}} \int_0^1 \left(\mathbb{E} \left[|Z|^{\tilde{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x| \right] \right)^{\frac{1}{\tilde{q}_1}} \\ &\quad \times \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k,s_1}, \dots, s_j\}} \left(\frac{\partial_{\theta} q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}}{q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}} \right)^{\tilde{q}_2} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\tilde{q}_2}} \\ &\quad \times \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k,s_1}, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0 + \frac{uh}{\sqrt{n\Delta_n}}, \sigma(\ell), \lambda_0}}{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_n}} \right)^{\tilde{q}_3} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{\tilde{q}_3}} dh \\ &\leq C \frac{|u|}{\sqrt{n}} (\mathbb{E}[|Z|^{\tilde{q}_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x])^{\frac{1}{\tilde{q}_1}} e^{-\frac{w}{\tilde{q}_2} \sqrt{\frac{\Delta_n}{n}}} \left(\frac{\lambda_n}{\lambda_0} \right)^{\frac{j}{\tilde{q}_2}} (e^{-\lambda_0 \Delta_n} \lambda_0^j)^{\frac{1}{\tilde{p}_1 \tilde{q}_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{\tilde{q}_3}} (1 + |x|^q). \end{aligned}$$

Finally, using Hölder's inequality with $q_1, q_2, q_3 > 1$ conjugate, Lemmas 6.2 and 6.4 where notice that $\sigma_1 = \sigma_2 = \frac{\sigma(\ell)^2}{\theta_n} (1 - e^{-2\theta_n \Delta_n})$, we get that for any $p_1, p_2 > 1$ conjugate,

$$S_3 \leq \frac{|w|}{\sqrt{n\Delta_n}} \int_0^1 \left(\mathbb{E} \left[|Z|^{q_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x| \right] \right)^{\frac{1}{q_1}}$$

$$\begin{aligned}
& \times \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{\partial_{\lambda} q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}}}{\frac{\theta_n, \sigma(\ell), \lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}}} \right)^{q_2} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{q_2}} \\
& \times \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_n, \sigma(\ell), \lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}}}{\frac{\theta_n, \sigma(\ell), \lambda_n}} \right)^{q_3} \middle| X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x \right] \right)^{\frac{1}{q_3}} dh \\
& \leq C \Delta_n^{\frac{1}{p_2 q_2} - \frac{1}{2}} \frac{|w|}{\sqrt{n}} \int_0^1 \left(\mathbb{E} \left[|Z|^{q_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x| \right] \right)^{\frac{1}{q_1}} e^{\frac{1}{q_2} \left(\frac{wh}{\sqrt{n\Delta_n}} - \frac{w}{\sqrt{n\Delta_n}} \right)} \left(\frac{\lambda_n}{\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}} \right)^{\frac{j}{q_2}} \\
& \quad \times \left(e^{-(\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}})\Delta_n} (\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}})^j \right)^{\frac{1}{p_1 q_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{q_3}} dh \\
& \leq C \frac{|w|}{\sqrt{n}} \int_0^1 \left(\mathbb{E} \left[|Z|^{q_1} |X_{t_k}^{\theta_n, \sigma(\ell), \lambda_n} = x| \right] \right)^{\frac{1}{q_1}} e^{\frac{1}{q_2} \left(\frac{wh}{\sqrt{n\Delta_n}} - \frac{w}{\sqrt{n\Delta_n}} \right)} \left(\frac{\lambda_n}{\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}}} \right)^{\frac{j}{q_2}} \\
& \quad \times \left(e^{-(\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}})\Delta_n} (\lambda_0 + \frac{wh}{\sqrt{n\Delta_n}})^j \right)^{\frac{1}{p_1 q_2}} (e^{-\lambda_n \Delta_n} \lambda_n^j)^{\frac{1}{q_3}} dh,
\end{aligned}$$

where p_2 and q_2 are chosen in order that $p_2 q_2 < 2$. This concludes the first inequality. The second inequality is proceeded similarly. Thus, the result follows. \square

6.2. Large deviation type estimates. For all $p \geq 1$, $k \in \{0, \dots, n-1\}$, and $i \in \{0, 1\}$, we set $\hat{J}_{\geq 2, k} := \{\Delta N_{t_{k+1}} \geq 2\}$, $\tilde{J}_{\geq 2, k} := \{\Delta M_{t_{k+1}} \geq 2\}$, and

$$\begin{aligned}
M_{i,p}^{\theta, \sigma, \lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{i,k}} \left((\Delta N_{t_{k+1}})^p - \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[(\Delta M_{t_{k+1}})^p | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] \right)^2 | X_{t_k} \right], \\
M_{\geq 2, p}^{\theta, \sigma, \lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{\geq 2, k}} \left((\Delta N_{t_{k+1}})^p - \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[(\Delta M_{t_{k+1}})^p | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] \right)^2 | X_{t_k} \right].
\end{aligned}$$

For jump diffusions [12, 13], a large deviation principle is used by conditioning on the jump structure to derive the large deviation type estimates in [12, Lemma 2.4] and [13, Lemma 5.4]. For the O-U process (1.1), the following analogue large deviation type estimates hold.

Lemma 6.6. *Let $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ such that $|\theta_0 - \theta| + |\lambda_0 - \lambda| \leq \frac{C_0}{\sqrt{n\Delta_n}}$, $|\sigma_0 - \sigma| \leq \frac{C_0}{\sqrt{n}}$, for some constant $C_0 > 0$. Then for all $p \geq 1$, $k \in \{0, \dots, n-1\}$, and n large enough, there exist constants $C, C_1 > 0$ such that for any $\alpha \in (0, \frac{1}{2})$, $p_1 > 1$, $q_1 > 1$ with $p_1 q_1 < 2$, and $\mu_1 \in (1, 2)$,*

$$M_{0,p}^{\theta, \sigma, \lambda} \leq C \left(e^{-C_1 \Delta_n^{2\alpha-1}} + \Delta_n^{\frac{2}{p_1 q_1}} \right), \quad (6.5)$$

$$M_{1,p}^{\theta, \sigma, \lambda} \leq C \left(e^{-C_1 \Delta_n^{2\alpha-1}} + \Delta_n^{\frac{2}{p_1 q_1}} \right), \quad (6.6)$$

$$M_{\geq 2, p}^{\theta, \sigma, \lambda} \leq C \Delta_n^{\frac{2}{\mu_1}}. \quad (6.7)$$

Proof. We start showing (6.5). Multiplying the random variable inside the conditional expectation of $M_{0,p}^{\theta, \sigma, \lambda}$ by $\mathbf{1}_{\tilde{J}_{0,k}} + \mathbf{1}_{\tilde{J}_{1,k}} + \mathbf{1}_{\tilde{J}_{\geq 2,k}}$ and applying Jensen's inequality, we get that

$M_{0,p}^{\theta,\sigma,\lambda} \leq 2(M_{0,1,p}^{\theta,\sigma,\lambda} + M_{0,2,p}^{\theta,\sigma,\lambda})$, where

$$\begin{aligned} M_{0,1,p}^{\theta,\sigma,\lambda} &:= \mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta,\sigma,\lambda} \left[\mathbf{1}_{\tilde{J}_{1,k}} |Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}| \right] | X_{t_k} \right], \\ M_{0,2,p}^{\theta,\sigma,\lambda} &:= \mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta,\sigma,\lambda} \left[\mathbf{1}_{\tilde{J}_{\geq 2,k}} (\Delta M_{t_{k+1}})^{2p} |Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}| \right] | X_{t_k} \right]. \end{aligned}$$

Proceeding as in Lemma 6.1 to change measures, applying Hölder's inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, Jensen's inequality, and Lemma 6.2 where notice that $\theta \rightarrow \theta_0$, $\sigma \rightarrow \sigma_0$ as $n \rightarrow \infty$, we get

$$\begin{aligned} M_{0,1,p}^{\theta,\sigma,\lambda} &= \frac{e^{\lambda\Delta_n}}{e^{\lambda_0\Delta_n}} \mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \frac{q_{(0)}^{\theta_0,\sigma_0,\lambda_0}}{q_{(0)}^{\theta,\sigma,\lambda}} (\Delta_n, X_{t_k}^{\theta,\sigma,\lambda}, X_{t_{k+1}}^{\theta,\sigma,\lambda}) \tilde{\mathbb{E}}_{X_{t_k}}^{\theta,\sigma,\lambda} [\mathbf{1}_{\tilde{J}_{1,k}} |Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}^{\theta,\sigma,\lambda}|] | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \\ &\leq e^{(\lambda-\lambda_0)\Delta_n} \left(\mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \left(\frac{q_{(0)}^{\theta_0,\sigma_0,\lambda_0}}{q_{(0)}^{\theta,\sigma,\lambda}} (\Delta_n, X_{t_k}^{\theta,\sigma,\lambda}, X_{t_{k+1}}^{\theta,\sigma,\lambda}) \right)^{p_1} | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \right)^{\frac{1}{p_1}} \\ &\quad \times \left(\mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta,\sigma,\lambda} [\mathbf{1}_{\tilde{J}_{1,k}} |Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}^{\theta,\sigma,\lambda}|] | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \right)^{\frac{1}{p_2}} \\ &\leq C \left(M_{0,1,1,p}^{\theta,\sigma,\lambda} \right)^{\frac{1}{p_2}}, \end{aligned}$$

for some constant $C > 0$ and n large enough. Here, using Bayes' formula, we get that

$$\begin{aligned} M_{0,1,1,p}^{\theta,\sigma,\lambda} &:= \mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta,\sigma,\lambda} [\mathbf{1}_{\tilde{J}_{1,k}} |Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}^{\theta,\sigma,\lambda}|] | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \\ &= \mathbb{E} \left[\mathbf{1}_{\tilde{J}_{0,k}} \frac{q_{(1)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}^{\theta,\sigma,\lambda}, X_{t_{k+1}}^{\theta,\sigma,\lambda}) e^{-\lambda\Delta_n} \lambda \Delta_n}{p^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}^{\theta,\sigma,\lambda}, X_{t_{k+1}}^{\theta,\sigma,\lambda})} | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \\ &= \int_{\mathbb{R}} \frac{q_{(1)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) e^{-\lambda\Delta_n} \lambda \Delta_n}{p^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y)} q_{(0)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) e^{-\lambda\Delta_n} dy. \end{aligned}$$

We next divide the dy integral into the subdomains $J_1 := \{y \in \mathbb{R} : |y - X_{t_k} e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1 - e^{-\theta\Delta_n})| > \Delta_n^\alpha\}$ and $J_2 := \{y \in \mathbb{R} : |y - X_{t_k} e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1 - e^{-\theta\Delta_n})| \leq \Delta_n^\alpha\}$, where $\alpha \in (0, \frac{1}{2})$, and call each integral $M_{0,1,1,1,p}^{\theta,\sigma,\lambda}$ and $M_{0,1,1,2,p}^{\theta,\sigma,\lambda}$. We start bounding $M_{0,1,1,1,p}^{\theta,\sigma,\lambda}$. By (6.1),

$$p^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) \geq q_{(1)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) e^{-\lambda\Delta_n} \lambda \Delta_n. \quad (6.8)$$

Then using (6.2), the equality $e^{-|x|^2} = e^{-\frac{|x|^2}{2}} e^{-\frac{|x|^2}{2}}$, valid for all $x \in \mathbb{R}$, and $1 - e^{-2\theta\Delta_n} \leq 2\theta\Delta_n$,

$$\begin{aligned} M_{0,1,1,1,p}^{\theta,\sigma,\lambda} &\leq \int_{J_1} q_{(0)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) dy \\ &= \int_{J_1} \frac{1}{\sqrt{\frac{\pi}{\theta}\sigma^2(1 - e^{-2\theta\Delta_n})}} \exp \left\{ -\frac{(y - X_{t_k} e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1 - e^{-\theta\Delta_n}))^2}{\frac{1}{\theta}\sigma^2(1 - e^{-2\theta\Delta_n})} \right\} dy \\ &\leq e^{-\frac{1}{4\sigma^2}\Delta_n^{2\alpha-1}} \int_{J_1} \frac{1}{\sqrt{\frac{\pi}{\theta}\sigma^2(1 - e^{-2\theta\Delta_n})}} \exp \left\{ -\frac{(y - X_{t_k} e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1 - e^{-\theta\Delta_n}))^2}{\frac{2}{\theta}\sigma^2(1 - e^{-2\theta\Delta_n})} \right\} dy \\ &\leq C e^{-\frac{1}{4\sigma^2}\Delta_n^{2\alpha-1}}, \end{aligned}$$

for some constant $C > 0$, since the dy integral is Gaussian and thus finite. Next, by (6.1),

$$p^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) \geq q_{(0)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y)e^{-\lambda\Delta_n}. \quad (6.9)$$

Then using (6.3), (6.4), and Fubini's theorem,

$$\begin{aligned} M_{0,1,1,2,p}^{\theta,\sigma,\lambda} &\leq e^{-\lambda\Delta_n} \lambda \Delta_n \int_{J_2} q_{(1)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) dy \\ &\leq \int_{t_k}^{t_{k+1}} \int_{J_2} \frac{\lambda}{\sqrt{\frac{\pi}{\theta}\sigma^2(1-e^{-2\theta\Delta_n})}} \exp \left\{ -\frac{(y - X_{t_k}e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1-e^{-\theta\Delta_n}) - e^{-\theta(t_{k+1}-s_1)})^2}{\frac{1}{\theta}\sigma^2(1-e^{-2\theta\Delta_n})} \right\} dy ds_1. \end{aligned}$$

Notice that $e^{-\theta\Delta_n} \leq e^{-\theta(t_{k+1}-s_1)} \leq 1$, for $s_1 \in [t_k, t_{k+1}]$. Thus, $e^{-\theta(t_{k+1}-s_1)} \geq \frac{1}{2}$ for n large enough. Moreover, on J_2 we have $|y - X_{t_k}e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1-e^{-\theta\Delta_n})| \leq \Delta_n^\alpha$. Therefore, using the inequality $|u+v|^2 \geq \frac{|u|^2}{2} - |v|^2$, valid for all $u, v \in \mathbb{R}$, together with $\alpha \in (0, \frac{1}{2})$, we deduce

$$\begin{aligned} &\left| y - X_{t_k}e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1-e^{-\theta\Delta_n}) - e^{-\theta(t_{k+1}-s_1)} \right|^2 \\ &\geq \frac{e^{-2\theta(t_{k+1}-s_1)}}{2} - \left| y - X_{t_k}e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1-e^{-\theta\Delta_n}) \right|^2 \geq \frac{1}{8} - \Delta_n^{2\alpha} \geq \frac{1}{10}, \end{aligned}$$

for n large enough. Therefore, using again the equality $e^{-|x|^2} = e^{-\frac{|x|^2}{2}} e^{-\frac{|x|^2}{2}}$, valid for all $x \in \mathbb{R}$, and the fact that $1 - e^{-2\theta\Delta_n} \leq 2\theta\Delta_n$, we obtain that for n large enough,

$$\begin{aligned} M_{0,1,1,2,p}^{\theta,\sigma,\lambda} &\leq e^{-\frac{1}{40\sigma^2}\Delta_n^{-1}} \int_{t_k}^{t_{k+1}} \int_{J_2} \frac{\lambda}{\sqrt{\frac{\pi}{\theta}\sigma^2(1-e^{-2\theta\Delta_n})}} \\ &\quad \times \exp \left\{ -\frac{(y - X_{t_k}e^{-\theta\Delta_n} + \frac{\lambda}{\theta}(1-e^{-\theta\Delta_n}) - e^{-\theta(t_{k+1}-s_1)})^2}{\frac{2}{\theta}\sigma^2(1-e^{-2\theta\Delta_n})} \right\} dy ds_1 \\ &\leq C e^{-\frac{1}{40\sigma^2}\Delta_n^{-1}}, \end{aligned}$$

for some constant $C > 0$, since the dy integral is Gaussian and thus finite. Hence,

$$M_{0,1,1,p}^{\theta,\sigma,\lambda} \leq C e^{-\frac{1}{40\sigma^2}\Delta_n^{2\alpha-1}},$$

This implies that for any $\alpha \in (0, \frac{1}{2})$, n large enough, and for some constants $C, C_1 > 0$,

$$M_{0,1,p}^{\theta,\sigma,\lambda} \leq C e^{-C_1\Delta_n^{2\alpha-1}}, \quad (6.10)$$

Next, we set $g(X_{t_{k+1}}) := \tilde{E}_{X_{t_k}}^{\theta,\sigma,\lambda} \left[\mathbf{1}_{\tilde{J}_{\geq 2,k}} (\Delta M_{t_{k+1}})^{2p} | Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}} \right]$. Applying Lemmas 6.1 and 6.2, and Hölder's inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, we obtain that for n large enough,

$$\begin{aligned} M_{0,2,p}^{\theta,\sigma,\lambda} &\leq \mathbb{E} [g(X_{t_{k+1}}) | X_{t_k}] = e^{(\lambda-\lambda_0)\Delta_n} \left\{ \mathbb{E} \left[g(X_{t_{k+1}}^{\theta,\sigma,\lambda}) \mathbf{1}_{\hat{J}_{0,k}} \frac{q_{(0)}^{\theta_0,\sigma_0,\lambda_0}}{q_{(0)}^{\theta,\sigma,\lambda}} | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda} \right)^j \int_{\Sigma_k^j} \mathbb{E} \left[g(X_{t_{k+1}}^{\theta,\sigma,\lambda}) \mathbf{1}_{\{\hat{J}_{j,k,s_1,\dots,s_j}\}} \frac{q_{(j)}^{\theta_0,\sigma_0,\lambda_0}}{q_{(j)}^{\theta,\sigma,\lambda}} | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] ds_1 \cdots ds_j \right\} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\mathbb{E} \left[\left| g(X_{t_{k+1}}^{\theta, \sigma, \lambda}) \right|^{p_1} \middle| X_{t_k}^{\theta, \sigma, \lambda} = X_{t_k} \right] \right)^{\frac{1}{p_1}} \left\{ \left(\mathbb{E} \left[\mathbf{1}_{\hat{J}_{0,k}} \left(\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta, \sigma, \lambda}} \right)^{p_2} \middle| X_{t_k}^{\theta, \sigma, \lambda} = X_{t_k} \right] \right)^{\frac{1}{p_2}} \right. \\
&\quad \left. + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda} \right)^j \int_{\Sigma_k^j} \left(\mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{j,k}, s_1, \dots, s_j\}} \left(\frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta, \sigma, \lambda}} \right)^{p_2} \middle| X_{t_k}^{\theta, \sigma, \lambda} = X_{t_k} \right] \right)^{\frac{1}{p_2}} ds_1 \cdots ds_j \right\} \\
&\leq C \left(\mathbb{E} \left[\left| g(X_{t_{k+1}}^{\theta, \sigma, \lambda}) \right|^{p_1} \middle| X_{t_k}^{\theta, \sigma, \lambda} = X_{t_k} \right] \right)^{\frac{1}{p_1}} \left\{ 1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0}{\lambda} \right)^j \frac{\Delta_n^j}{j!} (e^{-\lambda \Delta_n} \lambda^j)^{\frac{1}{p_2}} \right\} \\
&\leq C \left(\mathbb{P}(\tilde{J}_{\geq 2, k} | Y_{t_k}^{\theta, \sigma, \lambda} = X_{t_k}) \right)^{\frac{1}{p_1 q_1}} \left(\mathbb{E} \left[(\Delta M_{t_{k+1}})^{2p_1 q_2} | Y_{t_k}^{\theta, \sigma, \lambda} = X_{t_k} \right] \right)^{\frac{1}{p_1 q_2}} \\
&\leq C \Delta_n^{\frac{2}{p_1 q_1}} \Delta_n^{\frac{1}{p_1 q_2}} \leq C \Delta_n^{\frac{2}{p_1 q_1}},
\end{aligned}$$

where we have used the finiteness of the sum w.r.t. j , the Hölder's inequality with $\frac{1}{q_1} + \frac{1}{q_2} = 1$, and chosen p_1, q_1 in order that $p_1 q_1 < 2$. This, together with (6.10), concludes (6.5). Here

$$\frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta, \sigma, \lambda}} \equiv \frac{q_{(0)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(0)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}), \quad \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta, \sigma, \lambda}} \equiv \frac{q_{(j)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(j)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}; s_1, \dots, s_j).$$

We next show (6.6). As for the term $M_{0,p}^{\theta, \sigma, \lambda}$, multiplying the random variable inside the conditional expectation of $M_{1,p}^{\theta, \sigma, \lambda}$ by $\mathbf{1}_{\tilde{J}_{0,k}} + \mathbf{1}_{\tilde{J}_{1,k}} + \mathbf{1}_{\tilde{J}_{\geq 2,k}}$, using $\mathbf{1}_{\tilde{J}_{1,k}^c} = 1 - \mathbf{1}_{\tilde{J}_{1,k}}$ and Jensen's inequality, we have that $M_{1,p}^{\theta, \sigma, \lambda} \leq 2(M_{1,1,p}^{\theta, \sigma, \lambda} + M_{1,2,p}^{\theta, \sigma, \lambda})$, where

$$\begin{aligned}
M_{1,1,p}^{\theta, \sigma, \lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{1,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{\tilde{J}_{1,k}^c} | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] \middle| X_{t_k} \right], \\
M_{1,2,p}^{\theta, \sigma, \lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{1,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{\tilde{J}_{\geq 2,k}} (\Delta M_{t_{k+1}})^{2p} | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] \middle| X_{t_k} \right].
\end{aligned}$$

Observe that $M_{1,1,p}^{\theta, \sigma, \lambda} = M_{1,1,0,p}^{\theta, \sigma, \lambda} + M_{1,1,2,p}^{\theta, \sigma, \lambda}$, where

$$\begin{aligned}
M_{1,1,0,p}^{\theta, \sigma, \lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{1,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{\tilde{J}_{0,k}} | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] \middle| X_{t_k} \right], \\
M_{1,1,2,p}^{\theta, \sigma, \lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{1,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{\tilde{J}_{\geq 2,k}} | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] \middle| X_{t_k} \right].
\end{aligned}$$

Proceeding as in Lemma 6.1 to change the measures, applying Hölder's inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, Jensen's inequality, and Lemma 6.2 where notice that $\theta \rightarrow \theta_0, \sigma \rightarrow \sigma_0$ as $n \rightarrow \infty$, we get that for n large enough,

$$\begin{aligned}
M_{1,1,0,p}^{\theta, \sigma, \lambda} &= e^{(\lambda - \lambda_0) \Delta_n} \frac{\lambda_0}{\lambda} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\mathbf{1}_{\{\hat{J}_{1,k}, s_1\}} \frac{q_{(1)}^{\theta_0, \sigma_0, \lambda_0}}{q_{(1)}^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}^{\theta, \sigma, \lambda}, X_{t_{k+1}}^{\theta, \sigma, \lambda}; s_1) \right. \\
&\quad \left. \times \tilde{\mathbb{E}}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{\tilde{J}_{0,k}} | Y_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}}^{\theta, \sigma, \lambda} \right] \middle| X_{t_k}^{\theta, \sigma, \lambda} = X_{t_k} \right] ds_1 \\
&\leq C \left(M_{1,1,0,1,p}^{\theta, \sigma, \lambda} \right)^{\frac{1}{p_2}},
\end{aligned}$$

for some constant $C > 0$. Here, using Bayes' formula, we get that

$$\begin{aligned} M_{1,1,0,1,p}^{\theta,\sigma,\lambda} &:= \mathbb{E} \left[\mathbf{1}_{\hat{J}_{1,k}} \tilde{\mathbb{E}}_{X_{t_k}}^{\theta,\sigma,\lambda} \left[\mathbf{1}_{\tilde{J}_{0,k}} |Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}^{\theta,\sigma,\lambda}| \right] | X_{t_k}^{\theta,\sigma,\lambda} = X_{t_k} \right] \\ &= \int_{\mathbb{R}} \frac{q_{(0)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) e^{-\lambda \Delta_n}}{p^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y)} q_{(1)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) e^{-\lambda \Delta_n} \lambda \Delta_n dy. \end{aligned}$$

Again we divide the dy integral into the subdomains $J_1 := \{y \in \mathbb{R} : |y - X_{t_k} e^{-\theta \Delta_n} + \frac{\lambda}{\theta}(1 - e^{-\theta \Delta_n})| > \Delta_n^\alpha\}$ and $J_2 := \{y \in \mathbb{R} : |y - X_{t_k} e^{-\theta \Delta_n} + \frac{\lambda}{\theta}(1 - e^{-\theta \Delta_n})| \leq \Delta_n^\alpha\}$, where $\alpha \in (0, \frac{1}{2})$, and call each integral $M_{1,1,0,1,1,p}^{\theta,\sigma,\lambda}$ and $M_{1,1,0,1,2,p}^{\theta,\sigma,\lambda}$. In the same way the term $M_{0,1,1,1,p}^{\theta,\sigma,\lambda}$ was treated, using (6.8), (6.2), the equality $e^{-|x|^2} = e^{-\frac{|x|^2}{2}} e^{-\frac{|x|^2}{2}}$, valid for all $x \in \mathbb{R}$, we get that for any $\alpha \in (0, \frac{1}{2})$, n large enough, and for some constant $C > 0$,

$$M_{1,1,0,1,1,p}^{\theta,\sigma,\lambda} \leq \int_{J_1} q_{(0)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) dy \leq C e^{-\frac{1}{4\sigma^2} \Delta_n^{2\alpha-1}}.$$

Next, as for the term $M_{0,1,1,2,p}^{\theta,\sigma,\lambda}$, using (6.9), (6.3) and (6.4), we get that for n large enough,

$$M_{1,1,0,1,2,p}^{\theta,\sigma,\lambda} \leq e^{-\lambda \Delta_n} \lambda \Delta_n \int_{J_2} q_{(1)}^{\theta,\sigma,\lambda}(\Delta_n, X_{t_k}, y) dy \leq C e^{-\frac{1}{40\sigma^2} \Delta_n^{-1}},$$

for some constant $C > 0$. Therefore, we have shown that $M_{1,1,0,p}^{\theta,\sigma,\lambda}$ satisfies (6.10).

Proceeding as for the term $M_{0,2,p}^{\theta,\sigma,\lambda}$, we have that $M_{1,2,p}^{\theta,\sigma,\lambda} + M_{1,1,2,p}^{\theta,\sigma,\lambda} \leq C \Delta_n^{\frac{2}{p_1 q_1}}$, for all $p_1 > 1$, $q_1 > 1$ with $p_1 q_1 < 2$. This concludes (6.6). Finally, using Hölder's inequality with $\frac{1}{\mu_1} + \frac{1}{\mu_2} = 1$ and $\mu_1 \in (1, 2)$, and $P(\hat{J}_{\geq 2,k} | X_{t_k}) \leq (\lambda \Delta_n)^2$, it is easy to check that $M_{\geq 2,p}^{\theta,\sigma,\lambda} \leq C \Delta_n^{\frac{\mu_1}{2}}$, for some constant $C > 0$, which concludes (6.7). Thus, the result follows. \square

6.3. Change of measures. The following technical lemma is used several times in Lemmas 4.6 and 4.9.

Lemma 6.7. *For all $k \in \{0, \dots, n-1\}$,*

$$\mathbb{E} \left[\tilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] = -\frac{\ell w}{\sqrt{n \Delta_n}} \Delta_n.$$

Proof. Using Girsanov's theorem in Lemma 3.2, the independence between $\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)}$ and $\frac{d\hat{\mathbb{P}}}{d\hat{Q}_k^{\theta_n, \lambda(\ell), \theta_0, \lambda_0, \sigma_0}}$, and the fact that $\mathbb{E}[\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)}] = -\frac{\ell w}{\sqrt{n \Delta_n}} \Delta_n$, we obtain that

$$\begin{aligned} &\mathbb{E} \left[\tilde{\mathbb{E}}_{X_{t_k}}^{\theta_n, \sigma_0, \lambda(\ell)} \left[\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)} | Y_{t_{k+1}}^{\theta_n, \sigma_0, \lambda(\ell)} = X_{t_{k+1}} \right] | X_{t_k} \right] \\ &= \mathbb{E}_{\hat{Q}_k^{\theta_n, \lambda(\ell), \theta_0, \lambda_0, \sigma_0}} \left[\left(\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)} \right) \frac{d\hat{\mathbb{P}}}{d\hat{Q}_k^{\theta_n, \lambda(\ell), \theta_0, \lambda_0, \sigma_0}} | X_{t_k} \right] \\ &= \mathbb{E} \left[\tilde{M}_{t_{k+1}}^{\lambda(\ell)} - \tilde{M}_{t_k}^{\lambda(\ell)} \right] \mathbb{E}_{\hat{Q}_k^{\theta_n, \lambda(\ell), \theta_0, \lambda_0, \sigma_0}} \left[\frac{d\hat{\mathbb{P}}}{d\hat{Q}_k^{\theta_n, \lambda(\ell), \theta_0, \lambda_0, \sigma_0}} | X_{t_k} \right] = -\frac{\ell w}{\sqrt{n \Delta_n}} \Delta_n, \end{aligned}$$

since the second expectation is equal to 1. Thus, the result follows. \square

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